Mathematics for Economists: Lecture 2

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Content of Lecture 2

- In Lecture 1, a gentle introduction to a linear model of market equilibrium.
- This Lecture:
  1. Gaussian elimination via an example
  2. Economic application 1: input-output model (Leontieff)
  3. Linear equations without full rank
  4. Economic application 2: linear model of exchange (Gale)
  5. Connections from applications to other models

Gaussian Elimination

- Consider the following system of linear equations:

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 - x_4 &= 0 \\ -x_3 + 2x_4 &= 5 \end{aligned}$$

- Write this in matrix form:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \end{pmatrix}.$$

Gaussian Elimination

- The augmented matrix corresponding to this system is:

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right).$$

- Step 1: Eliminate all terms a_{j1} below the first *pivot* a_{11} (i.e. first non-zero element in column 1). In this case, just multiply the first row by $\frac{1}{2}$ and add to second row.

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right)$$

Gaussian Elimination

- Step 2: Eliminate all terms blow the second pivot: add the second row multiplied by $\frac{2}{3}$ to the third row.

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right)$$

Gaussian Elimination

- Step 3: Eliminate the term below the third pivot: add row 3 multiplied by $\frac{3}{4}$ to row 4.

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{array}\right)$$

- Step 4: At this point, the matrix is in *row echelon form*, i.e. each row starts with more zeros than the previous row.
- Last row reads: $\frac{5}{4}x_4 = 5$ or $x_4 = 4$. You can substitute this back to row 3 that says $\frac{4}{3}x_3 - x_4 = 0$ to get $x_3 = 3$. From row 2, $x_2 = 2$ and from row 1, $x_1 = 1$.
- In Gauss-Jordan elimination, one eliminates all elements above and below the pivot in these steps.
- This requires more eliminations, but avoids the substitutions in the last step

Gaussian Elimination: Second Example

- Consider another example:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 + x_3 &= 2 \\ x_2 + x_3 &= 1 \end{aligned}$$

- In matrix form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

Gaussian Elimination: Second Example

- Step 1:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

- Notice that now the second pivot is in row 3. Swapping rows 2 and 3 does not change the solution to an equation. So Step 2: Swap rows 2 & 3:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

- Now you can read $x_3 = 1$, $x_2 = 0$, and $x_1 = 1$.

Gaussian Elimination: General Principles

- Three *elementary row operations* that leave the solutions to systems of equations unchanged:
  1. Multiplying a row by a real number
  2. Adding rows to other rows
  3. Swapping rows
- Every matrix can be transformed to its row echelon form by elementary row operations.
- The *rank* of a matrix is the number of non-zero rows in its row echelon form.
- A linear system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution if and only if the rank of the coefficient matrix $\boldsymbol{A}$ is equal to the rank of the augmented matrix $(\boldsymbol{A}|\boldsymbol{b})$.
- If rank $(\boldsymbol{A})$ = rank $(\boldsymbol{A}|\boldsymbol{b})$ = n, the solution is unique, if rank $(\boldsymbol{A})$ = rank $(\boldsymbol{A}|\boldsymbol{b})$ ¡ n, then the system has infinitely many solutions.

Economic Application 1: Linear Input-Output Model

- Suppose (as Leontieff in the 1950's did) that an economy consists of s few aggregated sectors, for simplicity: Manufacturing, Agriculture and Services.
- If you have access to national accounts, you can compute the following: how much (say in monetary terms) each sector produces final consumer product denoted by $\boldsymbol{b} = (b_A, b_M, b_S)$.
- You can also compute how much each sector uses the products of the three sectors as intermediate goods or inputs in the production of the output:
  - For $i, j \in \{A, M, S\}$ denote by a_{ij} the amount of sector i product needed to produce one unit in sector j.
- Let's assume that production is linear:
  - To produce x_j units in sector j, you need $a_{ij}x_j$ units of sector i product.
- Can you describe this economy via a system of linear equations?

Linear Input-Output Model

- Let $\boldsymbol{x} = (x_A, x_M, x_S)$ denote the total production vector for all sectors.
- We have the basic accounting identities (e.g. here for agriculture):

$$x_A = a_{AA}x_A + a_{AM}x_M + a_{AS}x_S + b_A.$$

- On the left-hand side is the total agricultural production and on the right hand-side, we have the uses of those products as intermediate products needed in the other sectors and as final consumption.
- We have three simultaneous linear equations:

$$\begin{aligned} x_A &= a_{AA}x_A + a_{AM}x_M + a_{AS}x_S + b_A, \\ x_M &= a_{MA}x_A + a_{MM}x_M + a_{MS}x_S + b_M, \\ x_S &= a_{SA}x_A + a_{SM}x_M + a_{MS}x_S + b_S. \end{aligned}$$

Linear Input-Output Model

- Write in matrix form:

$$\begin{pmatrix} 1 - a_{AA} & -a_{AM} & -a_{AS} \\ -a_{MA} & 1 - a_{MM} & -a_{MS} \\ -a_{SA} & -a_{SM} & 1 - a_{SS} \end{pmatrix} \begin{pmatrix} x_A \\ x_M \\ x_S \end{pmatrix} = \begin{pmatrix} b_A \\ b_M \\ b_S \end{pmatrix}.$$

- Or more concisely as:

$$(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{b},$$

  where $\boldsymbol{I}$ is the $n \times n$ identity matrix, and $\boldsymbol{A}$ is the matrix of coefficients a_{ij}.
- The system has a unique solution for all $\boldsymbol{b}$ if rank $(\boldsymbol{I} - \boldsymbol{A}) = n$. But is this good enough? Shouldn't we also have

$$\boldsymbol{x} \geq 0?$$

Linear Input-Output Model: Bad Numerical Example

- Suppose that to produce $a_{ii} = 0$ and $a_{ij} = 1$ for $i \neq j$.
- Then we have the augmented matrix $(\boldsymbol{I} - \boldsymbol{A}|\boldsymbol{b})$:

$$\left(\begin{array}{rrr|r} 1 & -1 & -1 & b_A \\ -1 & 1 & -1 & b_M \\ -1 & -1 & 1 & b_S \end{array}\right)$$

- Elimination using the first pivot gives:

$$\left(\begin{array}{rrr|r} 1 & -1 & -1 & b_A \\ 0 & 0 & -2 & b_M + b_A \\ 0 & -2 & 0 & b_S + b_A \end{array}\right)$$

- You see immediately from the last two lines above that in any solution to the system, $x_M, x_S < 0$ for positive final consumptions and we conclude that $\boldsymbol{I} - \boldsymbol{A}$ is not a valid input-output matrix.

Linear Input-Output Model: Positive solutions

- Is there a reasonable condition that would guarantee the existence of positive solutions?
- We say that a matrix $\boldsymbol{D}$ is a *dominant diagonal* matrix if
  1. $d_{ii} > 0$ for all $i \in \{1, ..., n\}$,
  2. $d_{ij} \leq 0$ for all $i \neq j$,
  3. $\sum_{i=1}^{n} d_{ij} > 0$ for all $j \in \{1, ..., n\}$.
- The first condition just says that the production of any sector needs less of its own product as input than it gets as output.
- The second says that each sector produces a single output.
- The third condition means that each sector produces a positive value added (since we use the dollar values for inputs and outputs from the national accounts).

Linear Input-Output Model: Positive solutions

Proposition

If $\boldsymbol{D}$ is an $n \times n$ dominant diagonal matrix, then the equation system

$$\boldsymbol{Dx} = \boldsymbol{b}$$

has a unique solution $\boldsymbol{x} \geq 0$ for all $\boldsymbol{b} \geq 0$.

A proof is provided at the end of these lecture notes for those interested in seeing how these models work.

Linear Input-Output Model: Good Numerical Example

Suppose we have the the following input-output matrix:

$$\begin{pmatrix} x_A \\ x_M \\ x_S \end{pmatrix} = \begin{pmatrix} 0 & .2 & .6 \\ .3 & 0 & .1 \\ .5 & .4 & 0 \end{pmatrix} \begin{pmatrix} x_A \\ x_M \\ x_S \end{pmatrix} + \begin{pmatrix} b_A \\ b_M \\ b_S \end{pmatrix}.$$

Is it possible to produce $(b_A, b_m, b_S) = (1, 1, 1)$ The augmented matrix $(\boldsymbol{I} - \boldsymbol{A}|\boldsymbol{b})$ for this input-output system is:

$$\left(\begin{array}{ccc|c} 1 & -.2 & -.6 & 1 \\ -.3 & 1 & -.1 & 1 \\ -.5 & -.4 & 1 & 1 \end{array}\right)$$

Linear Input-Output Model: Good Numerical Example

Eliminating the first column gives:

$$\left(\begin{array}{ccc|c} 1 & -.2 & -.6 & 1 \\ 0 & .94 & .28 & 1.3 \\ 0 & -.5 & .7 & 1.5 \end{array}\right)$$

For numerical ease, eliminate the middle element in the third column with the last equation to get:

$$\left(\begin{array}{ccc|c} 1 & -.2 & -.6 & 1 \\ 0 & 1.14 & 0 & .7 \\ 0 & -.5 & .7 & 1.5 \end{array}\right)$$

Now you can solve: $x_M = \frac{.7}{1.14} = .61$, by substituting, you get $x_S = 2.58$, and $x_A = 2.67$

Input-Output Model: How to use it?

- So far we have talked about the quantities side of production.
- What about prices and value added?
- Let v_i be the value added per unit of production in sector i and $\boldsymbol{v} = (v_1, ..., v_n)$.
- Then v_i is the i^{th} element in the row vector $\boldsymbol{p}^\top(\boldsymbol{I} - \boldsymbol{A})$, where $\boldsymbol{p} = (p_1, ..., p_n) > 0$ is the price vector for the goods. (Exercise: Can you show that for each $v \geq 0$, such a price vector exists?)
- One fundamental identity for national accounts is that the total value added in the economy equals the value of final consumption or $\boldsymbol{v}^\top \boldsymbol{x} = \boldsymbol{p}^\top \boldsymbol{b}$.
- This follows from the fact that they both equal $\boldsymbol{p}^\top(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}$.

Economic Application 2: A Linear Model of Exchange

- Economics: What is the simplest imaginable model of international trade?
- Mathematics: Should we ever be interested in matrices without full rank?
- Imagine n countries.
- Country j spends fraction a_{ij} of its income on goods from country i.
- Let $x_i(t)$ be the income of country i in trading round t.
- No income enters the system from the outside and all income from round t is spent on goods from the n countries in round $t + 1$.

A Linear Model of Exchange

- If all income is spent, this means that $\sum_{i=1}^{n} a_{ij} = 1$ for all j.
- Let $\boldsymbol{x}(t) = (x_1(t), ..., x_n(t))$. Then we have:

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t),$$

  where $\boldsymbol{A}$ is the exchange matrix with ij^{th} element a_{ij}.
- Does a stable distribution of income exist?
- With this we ask if an $\boldsymbol{x} \neq 0$ exists such that:

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}.$$

A Linear Model of Exchange

- by writing the left hand side as $\boldsymbol{Ix}$, we see that this is the same having:

$$(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = 0.$$

- Non-zero solutions to a homogenous equation exist if and only if the matrix on the left hand side does *not* have full rank.
- Consider the following exchange matrix:

$$\boldsymbol{A} = \begin{pmatrix} .2 & .2 & .6 \\ .2 & .4 & .1 \\ .6 & .4 & .3 \end{pmatrix}.$$

- For the sake of some variety, let's check the rank of this matrix by computing its determinant
- $\det \boldsymbol{A} = .2(.12 - .04) - .2(.06 - .24) + .6(.02 - .24) = .016 + .036 - .132 \neq 0$ so it has full rank.

A Linear Model of Exchange

- What about $(\boldsymbol{I} - \boldsymbol{A})$?

$$(\boldsymbol{I} - \boldsymbol{A}) = \begin{pmatrix} .8 & -.2 & -.6 \\ -.2 & .6 & -.1 \\ -.6 & -.4 & .7 \end{pmatrix}.$$

- You can see that the third row is the sum of the first two rows multiplied by minus 1 and the rank is not full.
- You may recall from Matrix Algebra that we say that 1 is an eigenvalue of $\boldsymbol{A}$.
- If you eliminate the first column with the first pivot, you get:

$$\begin{pmatrix} .8 & -.2 & -.6 \\ 0 & .55 & -.25 \\ 0 & -.55 & .25 \end{pmatrix}.$$

A Linear Model of Exchange

- Eliminating using the second pivot gives the row echelon form:

$$\begin{pmatrix} .8 & -.2 & -.6 \\ 0 & .55 & -.25 \\ 0 & 0 & 0 \end{pmatrix}.$$

- This shows that any vector of the form $x_3(\frac{5}{44} + .75, \frac{5}{11}, 1)$ satisfies the homogenous equation.
- We say that $(\frac{5}{44} + .75, \frac{5}{11}, 1)$ is en eigenvector of $\boldsymbol{A}$.
- Since x_3 is arbitrary, it is often nice to normalize the incomes to sum to 1:

$$\boldsymbol{x} = (\frac{38}{102}, \frac{20}{102}, \frac{44}{102})$$

solves the equation.

Connections etc. for your information

- In week 6, we shall analyze the dynamics of $\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t)$.
- By repeated substitution, you see that $\boldsymbol{x}(k) = \boldsymbol{A}^k\boldsymbol{x}(0)$ so we see again that the key is to understand what happens to matrices when you raise them to powers.
- In Problem Set 1, you can relate this mathematical structure to popularity rankings.
- The most important real world application of this is Google Pagerank for ranking web sites.
- In that case, a_{ij} is the fraction of outward links from site j linking to i.
- There $\boldsymbol{x}$ solving $(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = 0$ is the vector of site ranks.

Next Lecture

- Non-linear economic models: utility functions and production functions
- Partial derivatives and total derivatives
- Derivative as a linear approximation of a non-linear function

Extra Material: Proof for Dominant Diagonal Matrices

Proposition

If $\boldsymbol{D}$ is an $n \times n$ dominant diagonal matrix, then the equation system

$$\boldsymbol{D}\boldsymbol{x} = \boldsymbol{b}$$

has a unique solution $\boldsymbol{x} \geq 0$ for all $\boldsymbol{b} \geq 0$.

Proof We prove this Proposition by induction on the size of the matrix. For $n = 1$ the claim is true by condition 1 for dominant diagonal matrices.

The induction hypothesis is that for all $(n-1) \times (n-1)$ dominant diagonal matrices $\boldsymbol{D}'$, the equation system $\boldsymbol{D}'\boldsymbol{x} = \boldsymbol{b}$ has a unique positive solution.

Extra Material: Proof for Dominant Diagonal Matrices

Need to show that if the claim is true for $(n-1) \times (n-1)$ dominant diagonal matrices, then it is also true for $n \times n$ dominant diagonal matrices.

In the first step, eliminate the first column of by using the first pivot $d_{11} > 0$.

After the elimination, the $(n-1) \times (n-1)$ submatrix $\boldsymbol{D}_{-11}$ consisting of rows and columns $\{2, ..., n\}$ is:

$$\boldsymbol{D}_{-11} = \begin{pmatrix} d_{22} - \frac{d_{21}}{d_{11}} d_{12} & \cdots & d_{2j} - \frac{d_{21}}{d_{11}} d_{1j} & \cdots & \cdots & d_{2n} - \frac{d_{21}}{d_{11}} d_{1n} \\ & & \vdots & & & \vdots \\ d_{i2} - \frac{d_{i1}}{d_{11}} d_{12} & \cdots & d_{jj} - \frac{d_{j1}}{d_{11}} d_{1j} & \cdots & \cdots & d_{in} - \frac{d_{i1}}{d_{11}} d_{1n} \\ \vdots & & \vdots & & & \vdots \\ d_{n2} - \frac{d_{n1}}{d_{11}} d_{12} & & d_{nj} - \frac{d_{n1}}{d_{11}} d_{1j} & & & d_{nn} - \frac{d_{n1}}{d_{11}} d_{1n} \end{pmatrix}.$$

Linear Input-Output Model: Positive solutions

We need to show that this is a dominant diagonal matrix, i.e. we need to verify conditions 1-3 for $\boldsymbol{D}_{-11}$.

The diagonal elements of D_{-11} are positive (exercise) and the off-diagonal elements are negative.
To prove the claim we need to verify that the sum of the elements in column j is positive for all $j \geq 2$.

Summing the elements in column j, we get:

$$\sum_{i=2}^{n}(d_{ij} - \frac{d_{i1}}{d_{11}} d_{1j}).$$

Since the original matrix D is by assumption a dominant diagonal matrix, we have:

$$\sum_{i=2}^{n} d_{ij} > -d_{1j}, \text{ and } \sum_{i=2}^{n} -\frac{d_{i1}}{d_{11}} d_{1j} > d_{1j}.$$

Hence the claim follows by summing.

Extra Material: Proof for Dominant Diagonal Matrices

For the augmented matrix $(\boldsymbol{D}|\boldsymbol{b})$, the rightmost column entries remain positive after the elimination step.

By the induction hypothesis, for each positive $\boldsymbol{b}'$, there is a unique positive vector $(x_2, ..., x_n)$ solving

$$\boldsymbol{D}_{-11}\boldsymbol{x} = \boldsymbol{b}'$$

.

Substituting into the first row of the original matrix $\boldsymbol{D}$ gives that x_1 is uniquely determined and positive. This proves the claim. $\square$

Extra Material on the Input-Output Model

- Consider the following reasoning: A final goods vector $\boldsymbol{b}$ generates intermediate goods demand for production $\boldsymbol{Ab}$.
- To produce the intermediate goods $\boldsymbol{Ab}$, generates production demand $\boldsymbol{A}(\boldsymbol{Ax})$.
- Continuing this line of thinking for K rounds you need $(\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + ... + \boldsymbol{A}^K)\boldsymbol{b}$ of resources to satisfy the final demand and all the intermediate stages.
- By multiplying out $(\boldsymbol{I} - \boldsymbol{A})(\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + ... + \boldsymbol{A}^K) = (\boldsymbol{I} - \boldsymbol{A}^{k+1})$, you see that $\lim_{K\to\infty} \sum_{k=0}^{K} \boldsymbol{A}^k = (\boldsymbol{I} - \boldsymbol{A})^{-1}$ if $\lim_{K\to\infty} \boldsymbol{A}^K = 0$.
- Notice that whenever $\lim_{K\to\infty} \boldsymbol{A}^K = 0$, we see immediately that all elements in $(\boldsymbol{I} - \boldsymbol{A})^{-1}$ are positive since all elements in $\boldsymbol{A}$ are positive.
- Hence also $\boldsymbol{x} = (\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{b}$ is positive for all positive $\boldsymbol{b}$.