1 Setting the scene: Economics and mathematics

Economics studies the allocation of scarce resources amongst competing ends. We may reasonably ask how we operationalize this definition.

1. What are the ways to allocate?

2. How to evaluate the results?

3. What do we mean by scarcity?

4. How can we formalize such questions?

In economics we adopt an individualistic approach: economic agents are autonomous decision makers that act in the pursuit of their own individual objectives or goals. In the standard approach, we assume that the agents do not make systematic mistakes in their choices. We acknowledge that they act within constraints. They also react to changes on their environment

Equilibrium analysis is used to guarantee the consistency of individual decisions. In competitive markets, this is formalized by an equilibrating price mechanism. In game theory, equilibrium arises from the assumed consistency of expectations and realized behavior.

This course introduces mathematical methods that allow us to present and analyze the problem of an individual economic agent and basic tools for equilibrium analysis.

1.1 Optimization problem

Economic agents (also called decision makers) have objectives summarized in their objective functions. Here is a partial list of the types of objective functions often encountered:

1. utility function of a consumer

2. profit function of a firm

3. total surplus for an economic planner

Autonomous decision making combined with individualistic goals means that the goal is to realize the maximal value of the objective function. Unfortunately, economic choices are constrained by scarcity and our agents cannot have and save the cake.

1.2 Endogenous and exogenous variables

Some basic methodological points should be kept in mind. First of all, economic models are not descriptions of an objective reality. They are chosen by the modeler to represent an important economic trade-off. The idea is to select the most important features of an economic situation and ignore the rest.

Every model has variables that are determined within the model. These are called the endogenous variables of the model. Interesting models have variables not determined within the model called the exogenous variables. Exogenous variables and parameters of the model are similar in nature. Their values are not affected by the endogenous variables of the model. For example, in models of consumer choice, optimal demands are the endogenous variables whereas price and available income are exogenous to the model.

1.3 Mathematical formulation of economic problems

In order to formulate the problem, we need the following ingredients:

- All conceivable choices x from a set X.
- Evaluation of alternatives: objective function $f : X \to \mathbb{R}$.
- Scarcity in the form of feasible set: $G \subset X$.
- Parameters and exogenous variables, $\{\alpha, \beta, ...\}$ determining f, G.
- Exogenous variables are not determined in the model.

For concreteness, let's consider some economic problems from Principles of Economics I :

1. Consumer choice between food and leisure
    - x_1 food consumption, x_2 leisure. $X = \{(x_1, x_2)|x_i \in \mathbb{R},\ \text{for } i \in \{1,2\}, x_1 \geq 0, 0 \leq x_2 \leq 24\}$.
    - Utility from x: $f(x;\alpha) = f(x_1, x_2; \alpha)$, where α is a preference parameter. $f(x_1, x_2) \geq f(y_1, y_2)$ if and only if (x_1, x_2) is at least as (y_1, y_2). If $f(x_1, x_2) = f(y_1, y_2)$, then (x_1, x_2) and (y_1, y_2) are on the same indifference curve.

- Feasible set: $p_1 x_1 \leq w(24 - x_2)$, where price of food p_1 and wages w are exogenous variables.

2. Best responses of player 1 in two player games:

   - Own action $x_1 \in X_1$ (row in the matrix).
   - Payoff from own action: $f(x_1; x_2)$, where x_2 is the exogenous variable (for best responses of 1, we just compute the payoff for all possible choices $x_2 \in X_2$, i.e. for all rows in the matrix).
   - In this context, no further feasibility constraint.
   - Of course when solving the game, x_2 becomes also an endogenous variable.

In general, we want to find the feasible alternative (or alternatives) that results in the highest value of the objective function amongst the feasible choices. I.e., find $x^* \in G$ such that $f(x^*) \geq f(x)$ for all $x \in G$. We write this problem as:

$$\max_{x \in G} f(x).$$

1.4 Mathematical structure

What kinds of variables are x, α, β? Most often we take them to be real numbers, or real vectors, but sometimes sometimes they can represent discrete choices (such as choosing the row in a matrix or choosing between a red and a blue car).

When does a solution exist? Weierstrass theorem (in the second part of this course) or other existence results that are just hinted at can be invoked to show existence.

The main emphasis in this course is on how to find a solution? We approach this usually with the help of calculus and the first part of this course is centered around multivariate calculus. Unfortunately, calculus is not of much help for discrete problems. In more advanced courses, you will see some tools for handling discrete problems.

Another question is whether the solution is unique? The concepts of concavity and convexity of the objective function provide the key for addressing this question.

Finally, one is interested in understanding how endogenous variables react to changes in exogenous variables. This is a key question for models in economics and it goes under the name of comparative statics. The implicit function theorem is the key tool for answering this question and it is one of our first goals in this course.

1.5 The language and vocabulary of mathematics:

1.5.1 Sets

A set is a collection of elements. Sets are denoted by capital letters: $X, Y, Z, ...$ Sets can be defined extensively by enumerating all its elements (difficult for large sets such as real numbers):

$$X = \{x_1, x_2, ..., x_K\},$$

or based on a property:

$$X = \{x \in \mathbf{R} \,| 0 \leq x < 1\}$$

Often used sets in this course include: real numbers, $\mathbb{R}$, integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, natural numbers $\mathbb{N}$ etc.

1.5.2 Vectors and Cartesian products

The Cartesian product of X and Y is a new set denoted by: $X \times Y$. It is defined as:

$$X \times Y = \{(x, y) \,| x \in X, y \in Y\}.$$

If $X = Y$, we write $X \times Y = X^2$. For example real plane vectors are elements in $\mathbb{R}^2$:

$$\mathbb{R}^2 = \{(x_1, x_2) \,| x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}.$$

Similarly n -dimensional real vectors, $x \in \mathbb{R}^n$:

$$\mathbb{R}^n = \{(x_1, ..., x_n) \,| x_1 \in \mathbb{R}, ..., x_n \in \mathbb{R}\}.$$

We denote vectors by boldface letters.

Consider $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

$$\boldsymbol{x} = \boldsymbol{y} \text{ if } x_i = y_i \text{ for all } i \in \{1, ..., n\}.$$

$$\boldsymbol{x} \geq \boldsymbol{y} \text{ if } x_i \geq y_i \text{ for all } i \in \{1, ..., n\} \text{ and}$$
$$\text{for some } i, x_i > y_i.$$

$$\boldsymbol{x} \gg \boldsymbol{y} \text{ if } x_i > y_i \text{ for all } i \in \{1, ..., n\}.$$

Note that unlike for real numbers, we do not have for all $\boldsymbol{x}, \boldsymbol{y}$

$$(\boldsymbol{x} \geq \boldsymbol{y}) \text{ or } (\boldsymbol{y} \leq \boldsymbol{x}) \text{ or both.}$$

Example 1. Consider an economy with k consumers and n goods. The vector $\boldsymbol{x}^1 = (x_1^1, ..., x_n^1)$ is the consumption vector of consumer 1, vector $\boldsymbol{x}^2 = (x_1^2, ..., x_n^2)$ is the consumption vector of 2 etc.

Consumer j receives utility $u^j(\boldsymbol{x}^j)$ from consumption vector $\boldsymbol{x}^j$.

Let $\overline{\boldsymbol{x}}_i$ be the total quantity of good i available in the economy. An allocation for the economy is a list $\boldsymbol{x} = \left(x^1, x^2, ..., x^k\right)$ of all consumption vectors. How many components do you have to specify for an allocation?

An allocation is feasible if for all goods $i \in \{1, ..., n\}$, we have:

$$\Sigma_{j=i}^k x_i^j \leq \overline{x}_i.$$

In words, an allocation is feasible if total consumption of each goods is no larger than total quantity available.

Allocation $\boldsymbol{y} = \left(y^1, ..., y^k\right)$ *Pareto-dominates* allocation $\boldsymbol{x} = \left(x^1, x^2, ..., x^k\right)$, if for all j,

$$u^j\left(\boldsymbol{y}^j\right) \geq u^j\left(\boldsymbol{x}^j\right).$$

Allocation $\boldsymbol{y}$ is *Pareto-efficient* if $\boldsymbol{y}$ is feasible and there exists no feasible allocation $\boldsymbol{x}$ Pareto-dominating $\boldsymbol{y}$.