Mathematics for Economists
ECON-C1000
Spring 2021
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Problem Set 1 (With typos corrected): Due April 29, 2021

1. Consider this simplest possible model of equilibrium in an industry. The demand is given by $q_d = a - bp$, and the supply by $q_s = c + dp$. Market clearing for a closed economy states: $q_s = q_d$. Assume that $a, b, c, d > 0$ and $a > b$.
   (a) After setting $q_s = q_d = q$, write the equation system for market equilibrium determination in terms of p, q.
   (b) Changing a by Δa changes the equilibrium price and quantity. Draw the figure depicting the equilibrium and consider this effect. Denote the changes by $\Delta q, \Delta p$. What is your guess for $\frac{\Delta q}{\Delta p}$?
   (c) Answer the same question but for changes in c.
   (d) Solve for these changes from the linear model.
   (e) If you see only a single(p, q) pair, you cannot say much about supply or demand. You just have a single observation. What king of data would you like to see if you want to find the supply curve? Give real-life examples of such data in a market of your choice.
2. Show that for an arbitrary matrix $\boldsymbol{A}$, the matrix $\boldsymbol{B} = \boldsymbol{A}^\top \boldsymbol{A}$ (where $\boldsymbol{A}^\top$ is the transpose of $\boldsymbol{A}$) is symmetric, i.e. that for all elements of $\boldsymbol{B}$, $b_{ij} = b_{ji}$.
3. A stochastic matrix is a square matrix whose the elements in each column sum up to 1. In other words, if $\boldsymbol{A}$ is a stochastic $n \times n$ -matrix, then $\sum_{i=1}^{n} a_{ij} = 1$ for all j.
   (a) Show that the matrix
   $$\boldsymbol{A} = \begin{pmatrix} 0.3 & 0.5 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 03 & 0.4 & 0.1 \end{pmatrix}$$
   has full rank, but the matrix
   $$(\boldsymbol{I} - \boldsymbol{A}) = \begin{pmatrix} -0.7 & 0.5 & 0.4 \\ 0.4 & -0.9 & 0.5 \\ 0.3 & 0.4 & -0.9 \end{pmatrix},$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix, does not have full rank.

(b) Find a vector $\boldsymbol{x} \neq 0$ such that $(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{x} = 0$.

(c) Show that if $\boldsymbol{A}$ is any stochastic matrix, then $(\boldsymbol{I} - \boldsymbol{A})$ and $(\boldsymbol{A} - \boldsymbol{I})$ do not have full rank.

4. Consider a class of 30 students. A sociologist wants to understand the social hierarchy in the class and asks each student to endorse at least one other student in the class. Based on the responses she designs a popularity ranking for the students. We want to see how this can be done using the tools of linear models.

   (a) Form a matrix of endorsements as follows: Identify each row and each column in a 30×30 -matrix $\boldsymbol{A}$ with a student. The students endorsed by student j form the j^{th} column of the matrix as follows. If j endorses n_j other students, set $a_{ij} = \frac{1}{n_j}$ if j endorsed i and 0 otherwise (no self-endorsements). What can you say about $\sum_{i=1}^{30} a_{ij}$?

   (b) What is the interpretation of $y_i := \sum_{j=1}^{30} a_{ij}$? Is y_i a good measure for the popularity of the students?

   (c) Maybe endorsements from popular students are more important for the ranking. To capture this idea, consider a ranking vector $\boldsymbol{x} = (x_1, ..., x_30)$ for the students. Require that the ranking of each student i is the weighted sum endorsements a_{ij} wighted by the popularity ranking of the endorsing student j. Write the linear model capturing this idea as:

   $$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}.$$

   Show that $(\boldsymbol{A} - \boldsymbol{I})$ does not have full rank. This means that there is a non-zero solution to the linear system.

   (d) You can take it on faith (or find a proof using either Brouwer's fixed-point theorem or Farkas' lemma) that a strictly positive solution exists. Normalize the solution $\boldsymbol{x}$ so that $\sum_{i=1} 30x_i = 1$. Show by an example that there can be many such non-zero solutions if the students can be divided into cliques that do not endorse students in other cliques.

   (e) (No question here, but just for your information) If you perturb the matrix $\boldsymbol{A}$ to be $\boldsymbol{A}' = (1 - \epsilon)\boldsymbol{A} + \epsilon\frac{1}{n}\mathbf{1}$, where $\mathbf{1}$ is a matrix whose elements are all equal to 1, then

   $$\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{x}, \sum_{i=1}^{30} x_i = 1$$

has a unique positive solution. A milder condition guaranteeing the uniqueness is that between any two students i, j, there is a chain of endorsements $i, k_1, k_2, k_3, ..., k_m, j$ so that $a_{ik_1}, a_{k_1k_2}, ..., a_{k_mj} > 0$

5. Consider the production function $Y(K, L) = A\big(\alpha K^\rho + (1-\alpha)L^\rho\big)^{\frac{1}{\rho}}$ called constant elasticity of substitution function or CES-function.

   (a) Calculate and interpret the partial derivatives of this function with respect to capital $K > 0$ and labor $L > 0$.

   (b) Show that
   $$\lim_{\rho \to 0} Y(K, L) = AK^\alpha L^{1-\alpha}.$$
   In words, show that the CES-function converges to the Cobb-Douglas -function. (Hint: consider $lnY(K, L)$ and use l'Hôpital's rule as $\rho \to 0$.)