Mathematics for Economists

Problem set 0

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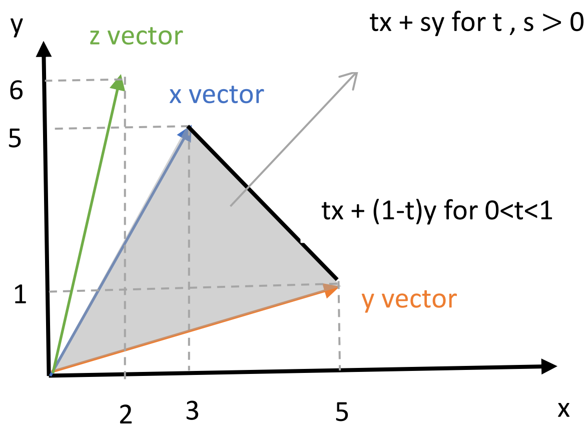
Question 1: vectors

a)

i)

x(3 , 5) , y(5 , 1)



$tx + (1-t)y$ is the line that attached two nodes (3 , 5) and (5 , 1)

$tx + sy$ for $s , y > 0$ is the space that is spanned by the summation of vectors x , and y, and since t and s are strictly positive the spanned area is between the two vectors.

since the vector z is not between the two vectors x and y, it is not possible to derive strictly positive s , t so that $z = tx + sy$

$$z = tx + sy \Rightarrow \begin{pmatrix} 3t + 5s \\ 5t + s \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \Rightarrow t = \frac{14}{11} \; and \; s = \frac{-4}{11}$$

ii)

grades = $(5 , 4 , 3 , 2 , 1)$

credits = $(6 , 3 , 6 , 3 , 3)$

we also define a vector where all the elements are equal to one:

$$ones(5) = (1 , 1 , 1 , 1 , 1)$$

$$total\ number\ of\ credits\ =\ credits\ .ones(5) = 6 + 3 + 6 + 3 + 3 = 21$$

now we construct the vector weights as:

$$weights = \ credits/total\ number\ of\ credits\ =\ (\frac{2}{7}, \frac{1}{7}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7})$$

and finally, GPA is equal to

$$GPA\ =\ grades\ .weights\ =\ \frac{23}{7}$$

iii)

$$income_i = \beta_1 age_i +\ \beta_2 yearsOfSchooling_i +\ \beta_3\ parentsIncome_i +\ \beta_4\ IQ_i + \cdots$$

b)

Using Pythagorean theorem, we have:

For two dimensional vectors:

$$||x|| = \sqrt{x_1^2 + x_2^2}$$

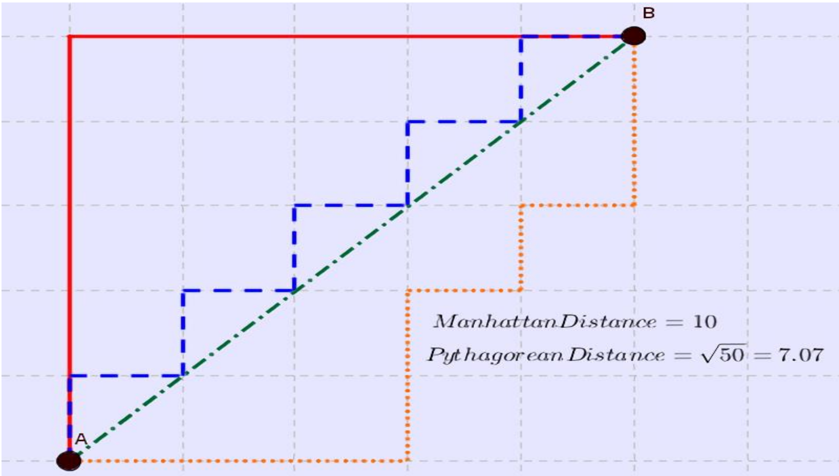
For a three dimensional vector:

$$||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

And finally for a k dimensional vector:

$$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_k^2}$$

c)



The green line shows the Pythagorean distance between points A and B. The Blue line, Orange line and the red line are equal and demonstrate the Manhattan distance between points A and B[1].

It is called the Manhattan distance because it is a distance a car would drive in a city where the buildings are laid out in square blocks and the straight streets intersects at right angles. In other words, it is a distance a car should take to get to his destinations in a city.

Sensible distance has the following characteristics:

$$d(x,y) \geq 0 \quad non-negativity$$
$$d(x,y) = 0 \iff \quad x = y$$
$$d(x,y) = d(y,x) \quad Symmetry$$
$$d(x,y) \leq d(x,z) + d(z,y) \quad \text{triangle inequality}$$

And Manhattan distance has all of these characteristics, so it is a sensible distance.

but the definition

$$d(x\,,y) \;=\; (\sqrt{x_1 - y_1} + \sqrt{x_2 - y_2})^2$$

is not a sensible distance because it does not have the third property. Moreover, if $x_1 < y_1 \; or \; x_2 < y_2$, the distance is going to be undefined.

[1] The figure is from: https://medium.com/@ozanerhansha/when-pi-doesnt-equal-3-14-73be75b2b90f

2. Matrices

a)

i)

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} \; and \; B = \begin{bmatrix} 5 & 3 \\ 7 & 2 \end{bmatrix}$$

$$C = A + B = \begin{bmatrix} 7 & 6 \\ 10 & 8 \end{bmatrix}$$

ii)

$$XY = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} 2*5+3*7 & 2*3+3*2 \\ 3*5+6*7 & 3*3+6*2 \end{pmatrix} = \begin{pmatrix} 31 & 12 \\ 57 & 21 \end{pmatrix}$$

$$YX = \begin{pmatrix} 5 & 3 \\ 7 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 5*2+3*3 & 5*3+3*6 \\ 7*2+2*3 & 7*3+2*6 \end{pmatrix} = \begin{pmatrix} 19 & 33 \\ 20 & 33 \end{pmatrix}$$

XY is not equal to YX, but sum of the diagonal elements for XY and YX are equal. It can be proved that it is true for any 2×2 matrices.

iii) It can be proved that A(BC) = (AB)C.

*I assumed that A, B and C are $n \times n$ matrices.

$$(A(BC))_{ij} = \sum_{k=1}^{n} A_{ik}(BC)_{kj} = \sum_{k=1}^{n} A_{ik} \sum_{r=1}^{n} B_{kr}C_{rj}$$

$$= \sum_{k}\sum_{r} A_{ik}B_{kr}C_{rj} = \sum_{r}\left(\sum_{k} A_{ik}B_{kr}\right)C_{rj} = \sum_{r}(AB)_{ir}C_{rj} = ((AB)C)_{ij}$$

b)

i) We have two 2×2 permutation matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and we have six 3×3 permutation matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

It can be proved that for the case of $n \times n$ matrices, we have $n!$ permutation matrices.

ii) Applied to matrix B, AB gives B with rows interchanged according to the permutation matrix A, and BA gives B with the columns interchanged according to the given permutation matrix, for example assume that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; and \; B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. we have

$$AB = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \; and \; BA = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Question 3:

First we write the equations in the matrix notation:

$$\begin{pmatrix} 2 & -3 & 3 \\ 2 & -1 & 2 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

Now we solve this system using the elementary row operations:

$$A = \begin{pmatrix} 2 & -3 & 3 \\ 2 & -1 & 2 \\ 3 & -2 & 1 \end{pmatrix}$$

And

$$b = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

So the augmented matrix is:

$$\left[\begin{array}{ccc|c} 2 & -3 & 3 & 1 \\ 2 & -1 & 2 & 3 \\ 3 & -2 & 1 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 3 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & \frac{5}{2} & \frac{-7}{2} & \frac{3}{2} \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -3 & 3 & 1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & \frac{-9}{4} & -1 \end{array}\right]$$

so $x_3 = \frac{4}{9}$ and inserting it in the second and the first equations we have:

$$x_2 = \frac{11}{9} \text{ and } x_1 = \frac{5}{3}$$

Question 4.

We discuss three different cases here. First is the Cournot competition where the competition is on quantity and the prices are endogenous in the model, and we can derive the price values from the demand function. This also means that the price values are endogenous to the consumers (using the demand function).

The second case is the Bertrand competition where the competition is on price. In here, the firms determine the price with their marginal costs so the price values are endogenous to the firms but they are exogenous to the consumers.

The last case is the discrete choice model of consumer choice , where each consumer has to choose between few choices, for example going to work with car or taking the bus. In this model the prices are exogenous (price of the ticket, price of fuel,…).

Question 5.

a) $f(x) = x^4 \Rightarrow f'(x) = 4x^3$
b) $f(x) = e^x \Rightarrow f'(x) = e^x$
c) $f(x) = (3x+2)^3 \Rightarrow f'(x) = 3 * 3 * (3x+2)^2 = 9 * (3x+2)^2$
d) $f(x) = \frac{4}{x^2+1} \Longrightarrow f'(x) = -\frac{4*2x}{(x^2+1)^2}$
e) $f(x) = x.\ln(x) \Rightarrow f'(x) = \ln(x) + 1$
f) $f(x) = x^x = e^{\ln(x^x)} = e^{x.\ln(x)} \Rightarrow f'(x) = (1+\ln(x)).e^{x.\ln(x)} = (1+\ln(x)).x^x$