COE-C3005 Finite Element and Finite difference methods

BAR-STRING MODELS

$$k\frac{\partial^2 a}{\partial x^2}+f'=m'\frac{\partial^2 a}{\partial t^2} \quad x\in\Omega\setminus I \quad \text{and} \quad [\![k\frac{\partial a}{\partial x}]\!]+F=0 \quad x\in I \quad t>0$$

$$a=\underline{a} \quad \text{or} \quad -n_x(k\frac{\partial a}{\partial x})+F=0 \quad x\in\partial\Omega \quad t>0,$$

$$a=g \quad \text{and} \quad \frac{\partial a}{\partial t}=h \quad x\in\Omega \quad t=0$$

PARTICLE SURROGATE METHOD

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i=m_i\ddot{a}_i \quad t>0$$

$$a_0=\underline{a}_0 \quad \text{or} \quad \frac{k}{\Delta x}(a_1-a_0)+F_0=m_0\ddot{a}_0 \quad \text{and} \quad a_n=\underline{a}_n \quad \text{or} \quad \frac{k}{\Delta x}(a_{n-1}-a_n)+F_n=m_n\ddot{a}_n \quad t>0$$

$$a_i=g_i \quad \text{and} \quad \dot{a}_i=h_i \quad t=0$$

FINITE DIFFERENCE METHOD

$$\frac{k}{\Delta x^2}(a_{i-1}-2a_i+a_{i+1})+f'=m'\ddot{a}_i \quad \text{or} \quad \frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F=0 \quad t>0$$

$$a_0=\underline{a}_0 \quad \text{or} \quad \frac{k}{\Delta x}(a_1-a_0)+F_0=0 \quad \text{and} \quad a_n=\underline{a}_n \quad \text{or} \quad \frac{k}{\Delta x}(a_{n-1}-a_n)+F_n=0 \quad t>0$$

$$a_i=g_i \quad \text{and} \quad \dot{a}_i=h_i \quad t=0$$

FINITE ELEMENT METHOD

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i+f'\Delta x=m'\frac{\Delta x}{6}(\ddot{a}_{i-1}+4\ddot{a}_i+\ddot{a}_{i+1}) \quad i\in\{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1-a_0)+F_0+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_0+\ddot{a}_1)=0 \quad \text{or} \quad a_0=\underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1}-a_n)+F_n+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_n+\ddot{a}_{n-1})=0 \quad \text{or} \quad a_n=\underline{a}_n,$$

$$a_i-g_i=0 \quad \text{and} \quad \dot{a}_i-h_i=0.$$

MEMBRANE MODEL

$$S'(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})+f'=m'\frac{\partial^2 w}{\partial t^2} \quad (x,y)\in\Omega \quad t>0$$

$$w=\underline{w} \quad \text{or} \quad S'(n_x\frac{\partial w}{\partial x}+n_y\frac{\partial w}{\partial y})=F' \quad (x,y)\in\partial\Omega \quad t>0$$

$$w=g \quad \text{and} \quad \frac{\partial w}{\partial t}=h \quad (x,y)\in\Omega \quad t=0$$

FINITE DIFFERENCE METHOD

$$\frac{S'}{h^2}[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)} \quad (i,j)\in I \quad t>0$$

$$w_{(i,j)}=0 \quad (i,j)\in\partial I \quad t>0$$

$$w_{(i,j)}-g_{(i,j)}=0 \quad \text{and} \quad \dot{w}_{(i,j)}-h_{(i,j)}=0 \quad (i,j)\in I \quad t=0$$

FINITE ELEMENT METHOD

$$S'[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+h^2 f'=$$

$$m'h^2\frac{1}{12}[\ddot{w}_{(i-1,j-1)}+\ddot{w}_{(i-1,j)}+\ddot{w}_{(i,j-1)}+6\ddot{w}_{(i,j)}+\ddot{w}_{(i+1,j)}+\ddot{w}_{(i,j+1)}+\ddot{w}_{(i+1,j+1)}] \quad (i,j)\in I \quad t>0$$

$$w_{(i,j)}=0 \quad (i,j)\in\partial I \quad t>0$$

$$w_{(i,j)}-g_{(i,j)}=0 \quad \text{and} \quad \dot{w}_{(i,j)}-h_{(i,j)}=0 \quad (i,j)\in I \quad t=0$$

SOLUTION METHODS

FOURIER SINE SERIES

$$\int_0^L \sin(k\pi\frac{x}{L})\sin(l\pi\frac{x}{L})dx=\frac{L}{2}\delta_{kl}$$

$$\alpha_k=\frac{2}{L}\int_0^L \sin(k\pi\frac{x}{L})a(x)dx \quad k\in\{1,2,\ldots\} \quad \Leftrightarrow \quad a(x)=\sum_{k\in\{1,2,\ldots\}}\alpha_k\sin(k\pi\frac{x}{L})$$

MODAL ANALYSIS AND MODE SUPERPOSITION

$$A(x)=\delta\sin(\lambda x)+\gamma\cos(\lambda x)\,, \quad \lambda=\omega\sqrt{\frac{m'}{k'}}$$

$$a(x,t) = \sum A_j[\frac{1}{\omega_j}\alpha_j \sin(\omega_j t) + \beta_j \cos(\omega_j t)]$$

$$\alpha_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x)h dx\,, \quad \beta_j = \frac{1}{\mathrm{A}_j^2}\int_\Omega A_j(x) g dx \text{ and } \mathrm{A}_j^2 = \int_\Omega A_j(x)A_j(x)dx\,.$$

MODAL ANALYSIS AND MODE SUPERPOSITION

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{A} = 0$$

$$\mathbf{a}(t) = \sum \mathbf{A}_j[\frac{1}{\omega_j}\alpha_j \sin(\omega_j t) + \beta_j \cos(\omega_j t)]$$

$$\alpha_j = \frac{1}{\mathrm{A}_j^2}\mathbf{A}_j^{\mathrm{T}}\mathbf{h}\,, \quad \beta_j = \frac{1}{\mathrm{A}_j^2}\mathbf{A}_j^{\mathrm{T}}\mathbf{g}\,, \quad \mathrm{A}_j^2 = \mathbf{A}_j^{\mathrm{T}}\mathbf{A}_j$$

CRANK-NICOLSON

$$\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i = \frac{1}{4+\alpha^2}\begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix} \text{ where } \alpha = \sqrt{\frac{k}{m}}\Delta t$$

$$\begin{bmatrix} \mathbf{I} & -\frac{1}{2}\mathbf{I} \\ \frac{\Delta t}{2}\mathbf{K} & \Delta t\mathbf{M} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{I} & \frac{1}{2}\mathbf{I} \\ -\frac{\Delta t}{2}\mathbf{K} & \mathbf{M} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$

DISCONTINUOUS-GALERKIN

$$\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i = \frac{2}{12+\alpha^4}\begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix} \text{ where } \alpha = \sqrt{\frac{k}{m}}\Delta t$$

$$\begin{bmatrix} \Delta t^2\mathbf{K} & -\frac{1}{2}\Delta t^2\mathbf{K} + \mathbf{M} \\ \frac{1}{2}\Delta t^2\mathbf{K} - \mathbf{M} & \mathbf{M} - \frac{1}{6}\Delta t^2\mathbf{K} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_i = \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix}\begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} \mathbf{a} \\ \dot{\mathbf{a}}\Delta t \end{Bmatrix}_0 = \begin{Bmatrix} \mathbf{g} \\ \mathbf{h}\Delta t \end{Bmatrix}$$