Mathematics for Economists
Aalto BIZ
Spring2021
Juuso Välimäki

Implicit functions: a single endogenous variable

Let's start with the most classic example of implicit functions. Recall from high school analytic geometry the equation defining the unit circle with center at the origin:

$$x^2 + y^2 = 1.$$

How does the y coordinate change if we change the x coordinate? A more careful phrasing of the question might ask 'How does y behave as a function of x around a point (x_0, y_0) on the unit circle?'

In this simple case, we could solve for y as a function of x:

$$y(x) = \pm\sqrt{1 - x^2}.$$

Two things stand out: fist of all $y(x)$ is not defined (at least as a real valued function) for $|x| > 1$. For $|x| \leq 1$ $y(x)$ is not a function since for each such x, there are two values for y that satisfy the equation of the circle.

If we resort to analyzing the behavior near an initial point (x_0, y_0), the sign of y_0 pick the 'branch' of $y(x)$ that we analyze. Let's compute then the derivative on the positive branch $y(x) = \sqrt{1 - x^2}$ at $x_0 = \frac{1}{2}$. Using the chain rule, we get:

$$y'(x_0) = \frac{1}{2}\frac{1}{\sqrt{1 - x_0^2}}{-2x_0} = -\frac{1}{\sqrt{3}}.$$

Consider next an alternative approach. Assume that we have a function $y(x)$ with $y(x_0) = y_0$ such that for $x \in B(x_0), \varepsilon)$, we have:

$$x^2 + y(x)^2 = 1.$$

In this case, we say that $y(x)$ is implicitly defined by the equation of the circle.

Since $x^2 + y(x)^2 = 1$ for all $x \in B(x_0), \varepsilon)$, we can take derivatives with respect to x on both sides of the equality:

$$2x + 2y(x)y'(x) = 0.$$

Evaluating at x_0, we have

$$y'(x_0) = -\frac{2x_0}{2y(x_0)} = -\frac{1}{\sqrt{3}}.$$

Notice the differences in the two approaches. In the first, we got an explicit function $y(x)$ near x_0. Of course this is great on principle, but try solving explicitly for y from:

$$x^2y^3 - 2xy^2 + 6y - 5x = 0.$$

The problem with the second comes in the sentence: 'Assume that we have a function $y(x_0)$ such that...'. Apart from that, the second approach is quite straightforward. Near x_0, y_0 we can compute the (small)$(\Delta x, \Delta y)$ such that $f(y_0, x_0) = f(y_0 + \Delta y, x_0 + \Delta x)$ from the linear approximation:

$$\frac{\partial f(y_0, x_0)}{\partial x}\Delta x + \frac{\partial f(y_0, x_0)}{\partial y}\Delta y = 0.$$

Solving for Δy from this gives:

$$\Delta y = -\frac{\frac{\partial f(y_0,x_0)}{\partial x}}{\frac{\partial f(y_0,x_0)}{\partial y}}\Delta x.$$

Therefore we have:

$$y'(x_0) = \frac{y(x_0 + \Delta x) - y(x_0)}{\Delta x} = \frac{\Delta y}{\Delta x} = -\frac{\frac{\partial f(y_0,x_0)}{\partial x}}{\frac{\partial f(y_0,x_0)}{\partial y}}.$$

As an example, let's consider the equation above:

$$f(y, x) = x^2y^3 - 2xy^2 + 6y - 5x = 0.$$

Observe that $(y_0, x_0) = (1, 1)$ solves the equation. Let's compute:

$$\frac{\partial f(y, x)}{\partial x} = 2xy^3 - 2y^2 - 5, \frac{\partial f(y, x)}{\partial y} = 3x^2y^2 - 4xy + 6.$$

Evaluating at $(1, 1)$, we have $\frac{\partial f(1,1)}{\partial x} = -5$, and $\frac{\partial f(1,1)}{\partial y} = 5$. Therefore $y'(1) = -\frac{-5}{5}$. In words, if we move x from $x_0 = 1$ to $1 + dx$, then y must also increase to $1 + dx$ for the equation $x^2y^3 - 2xy^2 + 6y - 5x = 0$ to remain true.

Let's see how the implicit function looks like when evaluated numerically:

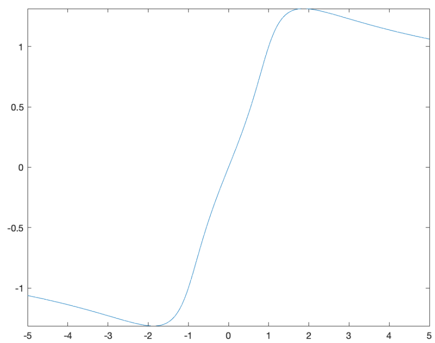

Figure 1: Implicit solution to $x^2y^3 - 2xy^2 + 6y - 5x = 0$.

This is of course quite easy in comparison to trying to find an explicit (local) solution to the equation around x_0. The remaining issue is that we need to determine when the implicit function $y(x_0)$ exists around x_0. Looking at the formula

$$y'(x_0) = -\frac{\frac{\partial f(y_0,x_0)}{\partial x}}{\frac{\partial f(y_0,x_0)}{\partial y}},$$

we see that at least we must have the denominator $\frac{\partial f(y_0,x_0)}{\partial y} =\neq 0$ for the right-hand side to make sense (to avoid dividing by zero). For the circle, the denominator vanishes at $y = 0$. It makes sense that we cannot have an implicit function $y(x)$ around $x_0 = 1$. There is no value of y to make satisfy the equation for $1 + h$ for $h > 0$. Implicit function theorem guarantees that this necessary condition for having an implicit function is also sufficient.

Theorem 1. Let $f(y, x)$ be a continuously differentiable in a neighborhood of (y_0, x_0) and $f(y_0, x_0) = 0$. If $\frac{\partial f(y_0,x_0)}{\partial y} \neq 0$, then there exists a continuously differentiable function $y(x)$ in a neighborhood B_{x_0} of x_0 such that:

1. $f(y(x), x) = 0$ for all $x \in B_{x_0}$,
2. $y(x_0) = y_0$,

3. The derivative of y at x_0 satisfies:

$$y'(x_0) = -\frac{\frac{\partial f(y_0,x_0)}{\partial x}}{\frac{\partial f(y_0,x_0)}{\partial y}}$$

The textbook has a proof of this theorem.

Implicit function theorem for many endogenous variables

Let's start with something that we already know from matrix algebra. Consider the system of equations:

$$\begin{aligned} a_{11}y_1 + ... + a_{1n}y_n + b_{11}x_1 + ... + b_{1m}x_m &= 0, \\ &\vdots \\ a_{n1}y_1 + ... + a_{nn}y_n + b_{n1}x_1 + ... + b_{nm}x_m &= 0. \end{aligned}$$

In matrix form:

$$\boldsymbol{A}\boldsymbol{y} + \boldsymbol{B}\boldsymbol{x} = 0,$$

where $\boldsymbol{A}$ on $n \times n$ matrix and $\boldsymbol{B}$ on $n \times m$ matrix, $\boldsymbol{y} = (y_1, ..., y_n)$, is a vector of endogenous variables and $\boldsymbol{x} = (x_1, ..., x_m)$ is a vector of exogenous variables.

Write this as:

$$\boldsymbol{f}(\boldsymbol{y}; \boldsymbol{x}) = 0.$$

Assume that

$$\boldsymbol{f}(\boldsymbol{y}_0; \boldsymbol{x}_0) = 0 \text{ or } \boldsymbol{A}\boldsymbol{y}_0 + \boldsymbol{B}\boldsymbol{x}_0 = 0,$$

and consider the effect of a small change $(d\boldsymbol{y}; d\boldsymbol{x}) = (dy_1, ..., dy_n; dx_1, ..., dx_m)$ on the value of :

$$\boldsymbol{f}(\boldsymbol{y}_0 + d\boldsymbol{y}, \boldsymbol{x}_0 + d\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y}_0, \boldsymbol{x}_0) = \boldsymbol{A}d\boldsymbol{y} + \boldsymbol{B}d\boldsymbol{x}$$

$$= D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{y}_0; \boldsymbol{x}_0)\, d\boldsymbol{y} + D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{y}_0; \boldsymbol{x}_0)\, d\boldsymbol{x},$$

where $D_{\boldsymbol{y}}(\boldsymbol{y_0}; \boldsymbol{x_0})$ consists of the partial derivatives of w.r.t. the endogenous variables $\boldsymbol{y}$ and $D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{y}_0; \boldsymbol{x}_0)$ w.r.t. the exogenous variables x.

For

$$\boldsymbol{f}\left(\boldsymbol{y};\boldsymbol{x}\right)=0.$$

to hold at $(\boldsymbol{y},\boldsymbol{x})=(\boldsymbol{y}_0+d\boldsymbol{y},\boldsymbol{x}_0+d\boldsymbol{x})$, the change must be zero:

$$D_{\boldsymbol{y}}\boldsymbol{f}\left(\boldsymbol{y}_0+d\boldsymbol{y},\boldsymbol{x}_0+d\boldsymbol{x}\right)d\boldsymbol{y}+D_x\boldsymbol{f}\left(\boldsymbol{y}_0+d\boldsymbol{y},\boldsymbol{x}_0+d\boldsymbol{x}\right)d\boldsymbol{x}=0.$$

In other words,

$$d\boldsymbol{y}=-D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{y}_0;\boldsymbol{x}_0)^{-1}D_{\boldsymbol{x}}(\boldsymbol{y_0};\boldsymbol{x_0})\mathbf{d}\boldsymbol{x}=\boldsymbol{A}^{\mathbf{-1}}\boldsymbol{B}\mathbf{d}\boldsymbol{x}.$$

If a single exogenous variable changes, then $\boldsymbol{B}d\boldsymbol{x}$ is a row vector and $d\boldsymbol{y}$ can be solved using Cramer's rule. This equation has a solution for all $d\boldsymbol{x}$ only if $\boldsymbol{A}^{-1}$ exists, i.e. if $\boldsymbol{A}=D_{\boldsymbol{y}}(\boldsymbol{y_0};\boldsymbol{x_0})$ has full rank. Let's see an example:

Example

$$\begin{aligned}2y_1+y_2+3x &= 0,\\ y_1-y_2-x &= 0.\end{aligned}$$

In matrix form:

$$\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}+\begin{pmatrix}3\\ -1\end{pmatrix}x=\begin{pmatrix}0\\ 0\end{pmatrix},$$

or

$$\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}-3\\ 1\end{pmatrix}x.$$

By Cramer's rule:

$$y_1=\frac{\det\begin{pmatrix}-3 & 1\\ 1 & -1\end{pmatrix}x}{\det\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}}=\frac{2}{-3}x,\, y_2=\frac{\det\begin{pmatrix}2 & -3\\ 1 & 1\end{pmatrix}x}{\det\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}}=\frac{5}{-3}x.$$

In other words, if dx is the change in the exogenous variable, then

$$dy_1=\frac{-2}{3}dx,\quad dy_2=\frac{-5}{3}dx.$$

This result can be generalized for the non-linear case in a neighborhood of $(\boldsymbol{y}_0;\boldsymbol{x}_0)$ and it is the main result of this section of the course: implicit function theorem for n endogenous and m exogenous variables.

Theorem 2. Let $\boldsymbol{f}(\boldsymbol{y},\boldsymbol{x})$ be a continuously differentiable in a neighborhood of $(\boldsymbol{y}_0,\boldsymbol{x}_0)$ such that $\boldsymbol{f}(\boldsymbol{y}_0;\boldsymbol{x}_0)=0$. If the matrix of partial derivatives with respect to the endogenous variables $D\boldsymbol{f}_{\boldsymbol{y}}(\boldsymbol{y}_0;\boldsymbol{y}_0)$ at $(\boldsymbol{y}_0;\boldsymbol{x}_0)$ has full rank, then there exists a continuously differentiable function $\boldsymbol{y}(\boldsymbol{x})$ in a neighborhood $B_{\boldsymbol{x}_0}$ of $\boldsymbol{x}_0$ such that:

1. $\boldsymbol{f}(\boldsymbol{y}(\boldsymbol{x});\boldsymbol{x})=0$ for all $\boldsymbol{x}\in B_{\boldsymbol{x}_0}$,
2. $\boldsymbol{y}(\boldsymbol{x}_0)=\boldsymbol{y}_0$,
3. The derivative of $\boldsymbol{y}$ at $\boldsymbol{x}_0$ satisfies:

$$D_{\boldsymbol{x}_0}\boldsymbol{y}(\boldsymbol{x}_0)=-D_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{y}_0;\boldsymbol{x}_0)^{-1}D_{\boldsymbol{x}}\boldsymbol{f}(\boldsymbol{y}_0;\boldsymbol{x}_0).$$

Proving this theorem is beyond the scope of this course. Let me just make some comments. Assuming properties 1. and 2. above, point 3. is an application of the chain rule in the vector-valued multivariate case. It is nothing more than a local version of the linear implicit function theorem. Parts 1. and 2. require some more sophisticated mathematics. Proving the existence of the implicit function $y(x)$ near $\hat{x}$ requires the use of a fixed point theorem (similar to the case of showing the existence of local solutions to differential equations). This is beyond the scope of this course.

Here is a computational example:

Example 1.

$$\boldsymbol{f}(\boldsymbol{y};\boldsymbol{x})=\begin{pmatrix} f_1(y_1,y_2;x_1,x_2) \\ f_2(y_1,y_2;x_1,x_2)\end{pmatrix}.$$

$$\begin{aligned} f_1(y_1,y_2;x_1,x_2) &= y_1y_2^2-x_1x_2+x_2+1=0, \\ f_2(y_1,y_2;x_1,x_2) &= y_1+\frac{x_1}{y_2}+x_2-5=0. \end{aligned}$$

Consider the system of equations in a neighborhood of the point

$$(\widehat{y}_1,\widehat{y}_2;\widehat{x}_1,\widehat{x}_2)=(1,1,2,2).$$

Check first that the equation is satisfied at $(1,1,2,2)$ and form the appropriate matrices of partial derivatives:

$$D_{\boldsymbol{y}}\boldsymbol{f}(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})=\begin{pmatrix} \frac{\partial f_1(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial y_1} & \frac{\partial f_1(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial y_2} \\ \frac{\partial f_2(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial y_1} & \frac{\partial f_2(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial y_2}\end{pmatrix}=\begin{pmatrix} \widehat{y}_2^2 & 2\widehat{y}_1\widehat{y}_2 \\ 1 & \frac{-\widehat{x}_1}{\widehat{y}_2^2}\end{pmatrix}=\begin{pmatrix} 1 & 2 \\ 1 & -2\end{pmatrix},$$

$$D_{\boldsymbol{x}}\boldsymbol{f}\left(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}}\right)=\begin{pmatrix}\frac{\partial f_1(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial x_1} & \frac{\partial f_1(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial x_2}\\ \frac{\partial f_2(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial x_1} & \frac{\partial f_2(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial x_2}\end{pmatrix}=\begin{pmatrix}-\widehat{x}_2 & 1-\widehat{x}_1\\ \frac{1}{\widehat{y}_2} & 1\end{pmatrix}=\begin{pmatrix}-2 & -1\\ 1 & 1\end{pmatrix}.$$

We see that $\det\left(D_{\boldsymbol{y}}\boldsymbol{f}\left(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}}\right)\right)\neq 0$, and therefore the matrix $D_{\boldsymbol{y}}\boldsymbol{f}(\hat{\boldsymbol{y}};\hat{\boldsymbol{x}})$ has full rank and an inverse matrix $[D_{\boldsymbol{y}}\boldsymbol{f}(\hat{\boldsymbol{y}};\hat{\boldsymbol{x}})]^{-1}$

Exercise: Show that

$$[D_{\boldsymbol{y}}\boldsymbol{f}(\hat{\boldsymbol{y}};\hat{\boldsymbol{x}})]^{-1}=\frac{-1}{4}\begin{pmatrix}-2 & -2\\ -1 & 1\end{pmatrix},$$

and therefore:

$$d\boldsymbol{y}=\frac{1}{4}\begin{pmatrix}-2 & -2\\ -1 & 1\end{pmatrix}\begin{pmatrix}-2 & -1\\ 1 & 1\end{pmatrix}d\boldsymbol{x}.$$

We could single out e.g. the effect of a change in x_1 on the endogenous variables near $(\hat{y}_1,\hat{y}_2,\hat{x}_1\hat{x}_2)=(1,1,2,2)$:

$$\begin{pmatrix}\frac{\partial f_1(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial y_1} & \frac{\partial f_1(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial y_2}\\ \frac{\partial f_2(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial y_1} & \frac{\partial f_2(\widehat{\boldsymbol{y}};\widehat{\boldsymbol{x}})}{\partial y_2}\end{pmatrix}\begin{pmatrix}dy_1\\ dy_2\end{pmatrix}+\begin{pmatrix}\frac{\partial f_1((\widehat{\boldsymbol{y}},\widehat{\boldsymbol{x}}))}{\partial x_1}\\ \frac{\partial f_2((\widehat{\boldsymbol{y}},\widehat{\boldsymbol{x}}))}{\partial x_1}\end{pmatrix}dx_1=0$$

Plugging in $(1,1,2,2)$, we get:

$$\begin{pmatrix}1 & 2\\ 1 & -2\end{pmatrix}\begin{pmatrix}dy_1\\ dy_2\end{pmatrix}+\begin{pmatrix}-2\\ 1\end{pmatrix}dx_1=0$$

Solving by Cramer's rule gives:

$$dy_1=\frac{\det\begin{pmatrix}2 & 2\\ -1 & -2\end{pmatrix}dx_1}{\det\begin{pmatrix}1 & 2\\ 1 & -2\end{pmatrix}}=\frac{1}{2}dx_1, dy_2=\frac{\det\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}}{\det\begin{pmatrix}1 & 2\\ 1 & -2\end{pmatrix}}=\frac{3}{4}dx_1.$$

We will return to more applications of the implicit function theorem in the part on constrained optimization.