Mathematics for Economists
ECON-C1000
Spring 2021
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Problem Set 2: Due May 6, 2021

1. Chain rule and implicit functions:
   (a) Let $f(x, y) = 6x^2y^2 + xy, x(t) = 10t^2 + 1$ and $y(t) = t^3 + 2t$. Compute the derivative of $f(x(t), y(t))$ with respect to t first by chain rule and then by plugging in the formulas for $x(t), y(t)$ and taking the derivative. Compare your solutions (and the ease of getting at the solutions).
   (b) Find the points where the gradient of the following functions is zero. We'll see next week how to decide if these points are minima or maxima.
      i. $f(x, y) = 8x^3 + 2xy - 3x^2 + y^2 + 1$
      ii. $f(x, y) = x + 2e^y - e^x - e^{2y}$
   (c) Consider the equation:
   $$f(y, x, z) = y^3x^2 + z^3 + yxz - 3 = 0.$$
   Treat y as the endogenous variable. Can you use implicit function theorem around $(x, y, z) = (1, 1, 1)$? Compute $\frac{dy}{dx}$ around this point.

2. Consider the system of equations:
$$\begin{aligned} \frac{\alpha}{y_1} - y_3z_1 &= 0 \\ \frac{\beta}{y_2} - y_3z_2 &= 0 \\ z_1y_1 + z_2y_2 - z_3 &= 0 \end{aligned}$$
   (a) Show that the system is satisfied at point
   $$(y_1, y_2, y_3, z_1, z_2, z_3) = (1, 1, 1, \alpha, \beta, \alpha + \beta)$$
   for $\alpha, \beta > 0$
   (b) Show that you can take (y_1, y_2, y_3) as endogenous variables and use the implicit function theorem there.

3. The profit π for a monopoly firms is computed as follows as a function of the output $q \geq 0$ of the firm:
$$\pi(q) = p(q)q - c(q),$$
where $p(q) = a - bq, a > 0, 0 < b < 1$ is the inverse demand function and $c(q) = \delta q^2$ is the cost function of the firm.

(a) To determine the optimal level of production for the firm, write the profit as a function of chosen output q and find the production level where the derivative with respect to q is zero. (Next week, we get the tools to see that this gives indeed the maximal profit).

(b) A tax $\tau \in (0, 1)$, per unit sold is set. The firm now receives $(1 - \tau)\, p$ in revenue for each of the units it sells. Determine (preferably without solving explicitly the new maximization problem) the effect of the tax on q.

4. Consider the following pair of functions for $x_1 > 0, x_2 > 0$:

$$f_1(x_1, x_2; c_1, c_2) = \frac{x_1}{x_1 + x_2} - c_1 x_1,$$

$$f_2(x_1, x_2; c_1, c_2) = \frac{x_2}{x_1 + x_2} - c_2 x_2,$$

where $c_1, c_2 > 0$.

(a) Compute for each $i \in \{1, 2\}$, $\frac{\partial f_i(x_1,x_2;c_1,c_2)}{\partial x_i}$. How do these partial derivatives depend on $\{x_1, x_2\}$?

(b) Form the system of equations where for each $i \in \{1, 2\}$, $\frac{\partial f_i(x_1,x_2;c_1,c_2)}{\partial x_i} = 0$. Solve the system.

(c) Set $c_1 = c_2 = c > 0$, and consider the modified pair of functions:

$$g_1(x_1, x_2; c, r) = \frac{x_1^r}{x_1^r + x_2^r} - c x_1,$$

$$g_2(x_1, x_2; c, r) = \frac{x_2^r}{x_1^r + x_2^r} - c x_2,$$

where $c > 0, 0 < r < 1$.

Form again the system where for each $i \in \{1, 2\}$, $\frac{\partial g_i(x_1,x_2;c,r)}{\partial x_i} = 0$. Determine the effect of r on the solutions $x_1(c, r), x_2(c, r)$ by using implicit function theorem (or alternatively solve explicitly).

(d) (Extra Credit) Show that a symmetric solution $x_1(c, r) = x_2(c, r) < 1/2$, exists for $0 < r < 1$ but not if r is very large. How does this relate to the discussion below?

(e) (For your information, no question here.) The story behind the problem is the following. Two countries $i \in \{1, 2\}$ are in conflict over a resource of size 1 and prepare for war over it. The outcome in the war depends on the sizes of the armies so that the probability of winning is proportional to the size of the armies: $\Pr\{\text{ i wins }\} = \frac{x_i}{x_1+x_2}$. The cost of raising an army of size x_i in country i is $c_i x_i$. Then

$$f_1(x_1, x_2; c_1, c_2) = \Pr\{i \text{ wins with army } x_i\} \times 1 - \text{ cost of } x_1.$$

The parameter r in the modification determines how sensitive the probability of winning is to differences in army size.