Mathematics for Economists
Aalto University School of Business
Spring 2021
JuusoVälimäki

Computing the derivative

In these notes, we go over the basic rules for computing the derivative of functions $f : \mathbb{R} \to \mathbb{R}$? These same rules apply for computing partial derivatives.

1. If $f(x) = a$ for all x, then $Df(\hat{x}) = 0$ for all $\hat{x}$.

2. If $f(x) = x$ then $Df(\hat{x}) = 1$ for all $\hat{x}$.

   (Linear homogeneity) If $g(x) = af(x)$, then $Dg(\hat{x}) = aDf(\hat{x})$.

3. (Additivity) Let $h(x) = f(x) + g(x)$. Then $Dh(\hat{x}) = Df(\hat{x}) + Dg(\hat{x})$.

4. (Product rule) Let
$$\phi(x) = f(x) g(x).$$
   Then
$$\begin{aligned} D\phi(\hat{x}) &= Df(x) g(x) = \lim_{h\to 0} \frac{\phi(\hat{x}+h) - \phi(\hat{x})}{h} \\ &= \lim_{h\to 0} \frac{f(\hat{x}+h) g(\hat{x}+h) - f(\hat{x}) g(\hat{x})}{h} \\ &= \lim_{h\to 0} \frac{(f(\hat{x}+h) - f(\hat{x})) g(\hat{x}+h) - f(\hat{x})(g(\hat{x}) - g(\hat{x}+h))}{h} \\ &= f'(\hat{x}) g(\hat{x}) + f(\hat{x}) g'(\hat{x}). \end{aligned}$$

5. (Chain rule) Let
$$\zeta(x) = g(f(x)).$$

Then

$$\begin{aligned}
D\zeta\left(\hat{x}\right) &= Dg\left(f\left(x\right)\right) = \lim_{h\rightarrow 0}\frac{\zeta\left(\hat{x}+h\right)-\zeta\left(\hat{x}\right)}{h} \\
&= \lim_{h\rightarrow 0}\frac{g\left(f\left(\hat{x}+h\right)\right)-g\left(f\left(\hat{x}\right)\right)}{h} \\
&= \lim_{h\rightarrow 0}\frac{g\left(f\left(\hat{x}\right)+f'\left(\hat{x}\right)h\right)-g\left(f\left(\hat{x}\right)\right)}{h} \\
&= \lim_{h\rightarrow 0}\frac{f'\left(\hat{x}\right)\left(g\left(f\left(\hat{x}+f'\left(\hat{x}\right)h\right)\right)-g\left(f\left(\hat{x}\right)\right)\right)}{f'\left(\hat{x}\right)h} \\
&= g'\left(f\left(\hat{x}\right)\right)f'\left(\hat{x}\right).
\end{aligned}$$

6. With these formulas, we can compute most derivatives that we need. For example, the derivative $D\phi(\hat{x})$ at $\hat{x}$ for the function

$$\phi\left(x\right) = x^2$$

is obtained from the product rule:

$$f\left(x\right) = g\left(x\right) = x.$$

We get

$$D\phi\left(\hat{x}\right) = \hat{x}+\hat{x} = 2\hat{x}.$$

By 'mathematical induction', we can see that for

$$f\left(x\right) = x^a,$$

$$Df\left(\hat{x}\right) = a\hat{x}^{a-1}.$$

By additivity and linear homogeneity, we can extend this to get the derivatives of all polynomial functions.

7. The rule for derivatives of quotients follows from the product rule. For $g(x) \neq 0$,

$$h\left(x\right) = \frac{f(x)}{g\left(x\right)}$$

can be written as:

$$h(x)g\left(x\right) = f\left(x\right).$$

Therefore

$$h'(\hat{x})\, g(\hat{x}) = f'(\hat{x}) - h(\hat{x})\, g'(\hat{x})$$

and therefore

$$h'(\hat{x}) = \frac{f'(\hat{x})}{g(\hat{x})} - \frac{f(\hat{x})\, g'(\hat{x})}{(g(\hat{x}))^2}.$$

8. The inverse function rule is a consequence of the chain rule: For all x, we have:

$$f^{-1}(f(x)) = x$$

Taking derivatives on both sides and denoting $y_0 = f(\hat{x})$:

$$Df^{-1}(y_0)\, Df(\hat{x}) = 1$$

and therefore:

$$Df^{-1}(y_0) = \frac{1}{Df(\hat{x})}.$$

9. A case that is not covered by the previous ones is the exponential function:

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \ldots$$

We will not proved this, but for convergent power series as the one above, we may differentiate element by element to get:

$$Df(\hat{x}) = 0 + 1 + \hat{x} + \frac{\hat{x}^2}{2} = \sum_{n=1}^{\infty} \frac{\hat{x}^{n-1}}{(n-1)!} = e^{\hat{x}}.$$

10. The logarithmic function denoted by $\ln(y)$ is the inverse function of the exponential function (defined for strictly positive y):

$$g(y) = \ln y,$$

$$f(x) = e^x,$$

$$g(f(x)) = x.$$

By chain rule:

$$Dg(y_0)\, Df'(\hat{x}) = 1, \text{ for all } \hat{x} \text{ and } y_0 = e^{\hat{x}}.$$

Therefore

$$Dg\left(y_0\right) = \frac{1}{Df\left(\hat{x}\right)} = \frac{1}{f\left(\hat{x}\right)} = \frac{1}{y_0}.$$

So we have:

$$D\ln y_0 = \frac{1}{y_0}.$$

11. Trigonometric functions etc. can be differentiated using their representations as power series or via direct limit arguments using basic identities from trigonometry.

12. l'Hôpital's rule is not a rule for computing derivatives but still closely connected. Suppose you want to evaluate for continuous functions f, g the following limit:

    $$\lim_{x\to\hat{x}} \frac{f(x)}{g(x)},$$

    but $f(\hat{x}) = g(\hat{x}) = 0$. In this case, you can use the fact that $f(x) = f'(\hat{x})(x-\hat{x}) + h.o.t.$ and $g(x) = g'(\hat{x})(x-\hat{x}) + h.o.t.$ to get:

    $$\frac{f(x)}{g(x)} = \frac{f'(\hat{x})(x-\hat{x}) + h.o.t.}{g'(\hat{x})(x-\hat{x}) + h.o.t.}.$$

    Dividing both the numerator and the denominator by $(x-\hat{x})$ and remembering that higher order terms vanish as $x \to \hat{x}$, we get:

    $$\lim_{x\to\hat{x}} \frac{f(x)}{g(x)} = \frac{f'(\hat{x})}{g'(\hat{x})},$$

    whenever one of these derivatives is non-zero. (If $g'(\hat{x}) = 0$) we take the limit to be infinite). This formula is called *l'Hôpital's rule*.

    If $f'(\hat{x}) = g'(\hat{x}) = 0$, you need to look at higher order derivatives. Taylor's theorem for higher order approximations is then useful.

13. Here is another rule for derivatives that is not used in this course, but may be useful later on:Leibniz' rule.

    Suppose that you have a function $g(x)$ defined via an integral:

$$g(x) = \int_0^{\beta(x)} f(y,x)dy$$

where $\beta(x)$ is a continuously differentiable function of x and f has a continuous partial derivative $\frac{\partial f(\hat{y},\hat{x})}{\partial x} < M$ for some $M < \infty$ and all y, x. Then

$$g'(\hat{x}) = \beta'(\hat{x})f(y,\beta(\hat{x})) + \int_0^{\beta(\hat{x})} \frac{\partial f(\hat{y},\hat{x})}{\partial x}dy.$$

The proof of this is covered in most books on advanced calculus. I will not write the proof, but here again, a picture is worth quite a few words.

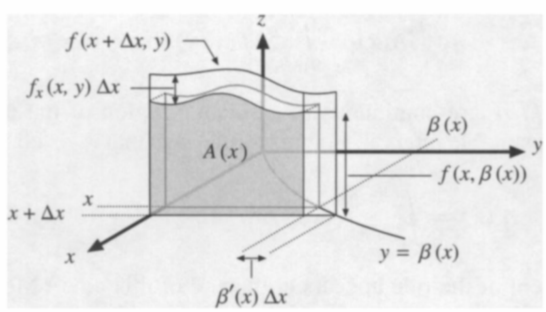

Figure 1: Leibniz rule.

It is an easy exercise to cover the case where the lower integration bound also depends on x.