Mathematics for Economists

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Solutions to the problem set 2:

Question 1

a)

First we try to obtain the derivative using the chain rule:

$$\frac{d}{dt}f(x,y) = \frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t)$$
$$= \left(12x(t)y^2(t) + y(t)\right) * (20t) + \left(12x^2(t)y(t) + x(t)\right) * (3t^2 + 2)$$
$$= \left(12(10t^2+1)(t^3+2t)^2 + (t^3+2t)\right) * (20t)$$
$$+ (12(10t^2+1)^2(t^3+2t) + 10t^2 + 1) * (3t^2+2)$$

Now by plugging x(t) and y(t) to f(x,y) and taking the derivative:

$$f\left(x(t), y(t)\right) = xy(1+6xy) = (10t^2+1)(t^3+2t)[1+6(10t^2+1)(t^3+2t)]$$
$$= (10t^5 + 21t^3 + 2t)[1 + 60t^5 + 126t^3 + 12t]$$
$$\Rightarrow f'(t) = (50t^4 + 63t^2 + 2)[1 + 60t^5 + 126t^3 + 12t] +$$
$$(10t^5 + 21t^3 + 2t)[300t^4 + 378t^2 + 12]=$$
$$(50t^4 + 63t^2 + 2)([1 + 120t^5 + 252t^3 + 24t + 1])$$

b)

$$f(x,y) = 8x^3 + 2xy - 3x^2 + y^2 + 1$$

$$\frac{df}{dx} = 0 \to 24x^2 + 2y - 6x = 0$$

$$\frac{df}{dy} = 0 \to 2x + 2y = 0 \to x = -y$$

using the last equation we have

$$24x^2 - 8x = 0 \to 8x(3x-1) = 0$$

so

$$x = 0\,, y = 0$$

$$x = \frac{1}{3}\,, y = -\frac{1}{3}$$

for the second equation:

$$f(x,y) = x + 2e^y - e^x - e^{2y}$$

$$\frac{df}{dx} = 0 \rightarrow 1 - e^x = 0 \rightarrow x = 0$$

$$\frac{df}{dy} = 0 \rightarrow 2e^y - 2e^{2y} = 0 \rightarrow e^y = e^{2y} \rightarrow y = 0$$

c)

Since y is the endogenous function, we can write:

$$f(x, y(x,z), z) = y^3(x,z)x^2 + z^3 + xy(x,z)z - 3 = 0$$

To use implicit function theorem, we need two conditions to be satisfied:

- Function f should be continuously differentiable at the point (1,1,1), which is clearly satisfied
- And $\frac{\partial f(\hat{x},\hat{y},\hat{z})}{\partial y} \neq 0$. To see if we have this condition:
  $\frac{\partial f(x,y,z)}{\partial y} = 3y^2x^2 + xz$ which is equal to 4 at $(1,1,1)$ so we can use the implicit function theorem here. Taking a derivative from function f to x we have:

$$f'(x,z) = \frac{\partial f}{\partial x} + y'_x \frac{\partial f}{\partial y} + z'_x \frac{\partial f}{\partial z}$$

But we know that the derivative of z with respect to x is equal to zero, so:

$$f'(x,z) = \frac{\partial f}{\partial x} + y'_x \frac{\partial f}{\partial y} = 0 \Rightarrow y'_x = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3}{4}$$

Question 2:

a) We have the system of equations:

$$\frac{\alpha}{y_1} - y_3 z_1 = 0$$
$$\frac{\beta}{y_2} - y_3 z_2 = 0$$
$$z_1 y_1 + z_2 y_2 - z_3 = 0$$

Now at the point:

$$(y_1, y_2, y_3, z_1, z_2, z_3) = (1,1,1,\alpha,\beta,\alpha+\beta)$$

We have:

$$\alpha - \alpha = 0$$
$$\beta - \beta = 0$$
$$\alpha + \beta - (\alpha + \beta) = 0$$

So the system is satisfied at this point.

b) Lets write the system of the equations in the following form:

$$f_1(y_1, y_2, y_3, z_1, z_2, z_3) = \frac{\alpha}{y_1} - y_3 z_1 = 0$$
$$f_2(y_1, y_2, y_3, z_1, z_2, z_3) = \frac{\beta}{y_2} - y_3 z_2 = 0$$
$$f_3(y_1, y_2, y_3, z_1, z_2, z_3) = z_1 y_1 + z_2 y_2 - z_3 = 0$$

And we know that the equations are satisfied at the point:

$$(y_1, y_2, y_3, z_1, z_2, z_3) = (1,1,1,\alpha,\beta,\alpha+\beta)$$

So we make the matrices of the partial derivatives:

$$D_y f(\hat{y},\hat{z}) = \begin{bmatrix} \frac{\partial f_1(\hat{y},\hat{z})}{\partial y_1} & \frac{\partial f_1(\hat{y},\hat{z})}{\partial y_2} & \frac{\partial f_1(\hat{y},\hat{z})}{\partial y_3} \\ \frac{\partial f_2(\hat{y},\hat{z})}{\partial y_1} & \frac{\partial f_2(\hat{y},\hat{z})}{\partial y_2} & \frac{\partial f_2(\hat{y},\hat{z})}{\partial y_3} \\ \frac{\partial f_3(\hat{y},\hat{z})}{\partial y_1} & \frac{\partial f_3(\hat{y},\hat{z})}{\partial y_2} & \frac{\partial f_3(\hat{y},\hat{z})}{\partial y_3} \end{bmatrix} = \begin{bmatrix} -\frac{\alpha}{\hat{y}_1^2} & 0 & -\hat{z}_1 \\ 0 & -\frac{\beta}{\hat{y}_2^2} & -\hat{z}_2 \\ \hat{z}_1 & \hat{z}_2 & 0 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & -\alpha \\ 0 & -\beta & -\beta \\ \alpha & \beta & 0 \end{bmatrix}$$

$$\det\left(D_y f(\hat{y},\hat{z})\right) = -\alpha\beta(\alpha+\beta)$$

And since

$$\alpha, \beta > 0$$

$$\det\left(D_y f(\hat{y},\hat{z})\right) \neq 0$$

So we have the necessary condition to use the implicit function theorem to obtain $\frac{dy}{dz}$.

Now we have:

$$dy = [D_y f(\hat{y},\hat{z})]^{-1}[D_z f(\hat{y},\hat{z})]dz$$

Question 3:

$$\pi(q) = p(q)q - c(q)$$

Where p(q) is the inverse demand function and is equal to:

$$p(q) = a - bq$$

And c(q) is the cost function of the firm and it is equal to:

$$c(q) = \delta q^2$$

a)

$$\pi(q) = p(q)q - c(q) = q(a - bq) - \delta q^2$$
$$\frac{d\pi}{dq} = a - 2bq - 2\delta q = 0 \Rightarrow q^* = \frac{a}{2(b+\delta)}$$

b) Now we have the per unit tax, so the profit function is:

$$\pi(q) = (1-\tau)p(q)q - c(q) = (1-\tau)q(a-bq) - \delta q^2$$
$$\frac{d\pi}{dq} = (1-\tau)(a-2bq) - 2\delta q = 0$$

using the implicit function theorem, we have:

$$f = (1-\tau)(a-2bq) - 2\delta q = 0$$

and the partial derivative of q with respect to τ is:

$$\frac{\partial q}{\partial \tau} = -\frac{\frac{\partial f}{\partial \tau}}{\frac{\partial f}{\partial q}} = (-1)\frac{a - 2bq}{2b(1-\tau) + 2\delta}$$

It is obvious that the denominator is always positive since $0 < \tau < 1$, but what can we say about the numerator $a - 2bq$? what is the revenue of the firm after selling q?

$$Revenue = p.q = (1-\tau)q(a - bq) \rightarrow MR = (1-\tau)(a - 2bq) > 0$$

And we know that the marginal revenue of the firm in operation is always positive. Consequently, the partial derivative of q with respect to τ is always negative so the amount of the optimal production will decrease as the result of the tax implementation.

Question 4:

a)

$$f_1(x_1, x_2; c_1, c_2) = \frac{x_1}{x_1 + x_2} - c_1 x_1$$

$$f_2(x_1, x_2; c_1, c_2) = \frac{x_2}{x_1 + x_2} - c_2 x_2$$

we first compute the partial derivatives with respect to x_1, x_2:

$$\frac{\partial f_1}{\partial x_1} = \frac{x_1 + x_2 - x_1}{(x_1 + x_2)^2} - c_1 = \frac{x_2}{(x_1 + x_2)^2} - c_1$$

$$\frac{\partial f_2}{\partial x_2} = \frac{x_2 + x_1 - x_2}{(x_1 + x_2)^2} - c_2 = \frac{x_1}{(x_1 + x_2)^2} - c_2$$

b)

the system of equations:

$$\frac{x_2}{(x_1 + x_2)^2} - c_1 = 0$$

$$\frac{x_1}{(x_1 + x_2)^2} - c_2 = 0$$

we can easily write the above equations as follows:

$$(x_1 + x_2)^2 = \frac{x_2}{c_1}$$

$$(x_1 + x_2)^2 = \frac{x_1}{c_2}$$

so

$$\frac{x_2}{c_1} = \frac{x_1}{c_2} \rightarrow x_2 = \frac{x_1 c_1}{c_2}$$

putting it into the second equation ($\frac{\partial f_2}{\partial x_2} = 0$), we will have:

$$\frac{x_1}{(x_1+\frac{x_1c_1}{c_2})^2} = c_2 \to \frac{1}{x_1(1+\frac{c_1}{c_2})^2} = c_2$$

$$x_1 = \frac{1}{c_2(\frac{c_1+c_2}{c_2})^2} \to x_1 = \frac{c_2}{(c_1+c_2)^2}$$

and

$$x_2 = \frac{c_1}{(c_1+c_2)^2}$$

c)

$$g_1 = \frac{x_1^r}{x_1^r + x_2^r} - cx_1$$

$$g_2 = \frac{x_2^r}{x_1^r + x_2^r} - cx_2$$

deriving the partial derivative with respect to x_1, we have:

$$\frac{\partial g_1}{\partial x_1} = \frac{(rx_1^{r-1})(x_1^r + x_2^r) - (rx_1^{r-1})x_1^r}{(x_1^r + x_2^r)^2} = \frac{(rx_1^{r-1})x_2^r}{(x_1^r + x_2^r)^2}$$

so the system of the equations is:

$$\frac{\partial g_1}{\partial x_1} = \frac{(rx_1^{r-1})x_2^r}{(x_1^r + x_2^r)^2} - c = 0$$

$$\frac{\partial g_2}{\partial x_2} = \frac{(rx_2^{r-1})x_1^r}{(x_1^r + x_2^r)^2} - c = 0$$

c,d)

$$g_1 = \frac{x_1^r}{x_1^r + x_2^r} - cx_1$$

$$g_2 = \frac{x_2^r}{x_1^r + x_2^r} - cx_2$$

$$\frac{\partial g_1}{\partial x_1} = \frac{rx_1^{r-1}(x_1^r + x_2^r) - rx_1^{r-1}x_1^r}{(x_1^r + x_2^r)^2} - c = \frac{rx_1^{r-1}x_2^r}{(x_1^r + x_2^r)^2} - c$$

so the partial derivatives are:

$$\frac{\partial g_1}{\partial x_1} = \frac{rx_1^{r-1}x_2^r}{(x_1^r + x_2^r)^2} - c = 0$$

$$\frac{\partial g_2}{\partial x_2} = \frac{rx_2^{r-1}x_1^r}{(x_1^r + x_2^r)^2} - c = 0$$

by solving the derived system of equations, we will have:

$$x_1^{r-1}x_2^r = x_2^{r-1}x_1^r \to x_1 = x_2$$

putting it in either of the equations, we get:

$$\frac{rx_1^{2r-1}}{4x_1^{2r}} = c \to \frac{r}{4x_1} = c \to x_1 = \frac{r}{4c}$$

and

$$x_1 = x_2 = \frac{r}{4c}$$

so

$$cx = \frac{r}{4} < \frac{1}{2}$$

e)

We know that the payoff of going to the war for each country is:

$$\pi(x_1, x_2, c_1, c_2) = \frac{x_i}{x_1 + x_2} - cx_i$$

since $x_1 = x_2$, the first part is equal to $\frac{1}{2}$ and obviously the second part which is the cost of going to the war shouldn't be greater than $\frac{1}{2}$,otherwise they have no intentions to do that.

$$\pi = \frac{1}{2} - cx$$

Now consider the case where $r > 2$. consequently $cx = \frac{r}{4} > \frac{1}{2}$ and the payoff of going to the war (with any amount of army size) will be negative for the countries.

It will be really useful to plot the two cases of the profit function for firm 1. Assume that firm 2 behave according to the equilibrium $x_2 = \frac{r}{4c}$ and c=0.005 and we plot the profit function for two values of r, r=0.5 and r=4.

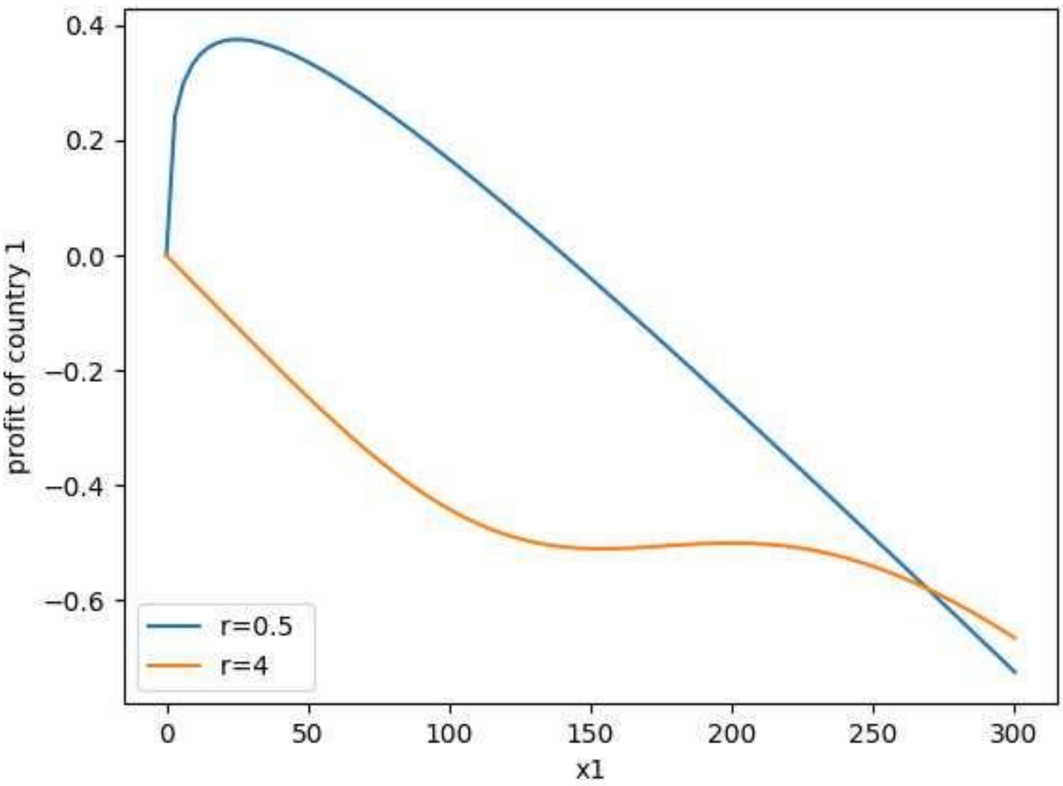

Figure 1

As you see in Figure 1, for $0 < r < 1$ the function is convex and it is easy to obtain the x_1^*. In the second case where $r > 2$, the function f is decreasing at first and it has a local maxima which brings us the negative profit.