Mathematics for Economists

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Solutions to the problem set 1

Question 1:

a) Demand function: $q^d = a - bp$

Supply function: $q^s = c + dp$

Market clearing condition implies that:

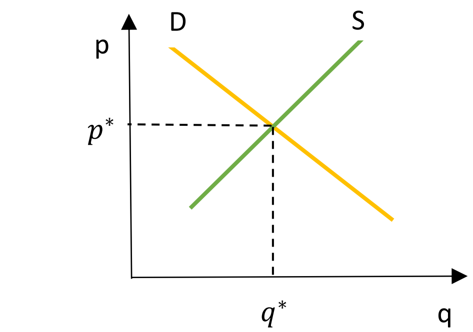
$$q^d = q^s = q$$

Consequently:

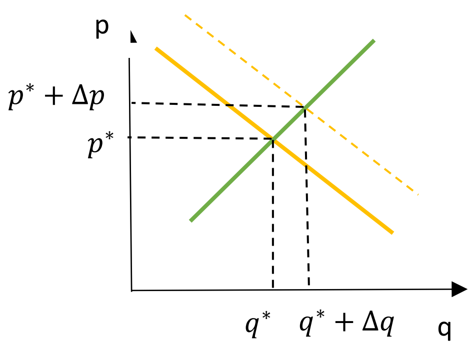
$$q = a - bp = c + dp \Rightarrow a - c = p(b + d) \Rightarrow p^* = \frac{a - c}{b + d}$$

$$q^* = a - b\left(\frac{a - c}{b + d}\right) = \frac{ab + ad - ba + bc}{b + d} = \frac{ad + bc}{b + d}$$

b)

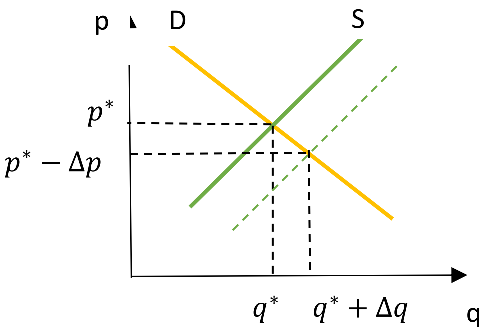


The equilibrium point from the market clearing condition



Increasing the parameter a is equivalent to the upward shift in the demand function. The amount of the shift is equal to Δa. Because of this shift, the Equilibrium price and the Equilibrium quantity will be shifted respectively Δp and Δq.

c)



On the other hand, increasing the parameter c is equivalent to shifting the supply function. As the result, the equilibrium price and the Equilibrium quantity will shift as well.

d)

We first solve the problem for the change in a:

$$a \to a + \Delta a \;\Rightarrow p + \Delta p = \frac{a + \Delta a - c}{b + d} \Rightarrow \;\Delta p = \frac{\Delta a}{b + d}$$

$$q + \Delta q = \frac{(a + \Delta a)d + bc}{b + d} \Rightarrow \Delta q = \frac{\Delta ad}{b + d}$$

And finally:

$$\frac{\Delta q}{\Delta p} = d$$

Now for the change in c:

$$c \to c + \Delta c \;\Rightarrow p + \Delta p = \frac{a - c - \Delta c}{b + d} \Rightarrow \;\Delta p = \frac{-\Delta c}{b + d}$$

$$q + \Delta q = \frac{ad + b(c + \Delta c)}{b + d} \Rightarrow \Delta q = \frac{b\Delta c}{b + d}$$

So:

$$\frac{\delta q}{\delta p} = -b$$

e)

Well we can not say a lot about supply curve if we observe only one pair of price and quantity, but if we have enough data (assume that you observe the market for a period of time), then we will see demand shifters as a result of what happens in the market. For example: lets go back to few months ago at the start of the pandemic. Obviously, there was an enormous shift in the demand of the hand washing soap or hand sanitizer. Assuming that the supply of these products are constant and it is not possible to increase it substantially at a short period of time, We can easily come up with an estimate for the shape of the supply function of these products.

Question 2:

We can prove the following properties for the transpose of a matrix:

1) $(A^T)^T = A$
2) $(A + B)^T = A^T + B^T$
3) $(kA)^T = kA^T$
4) $(AB)^T = B^T A^T$ (to prove it just compare the (i, j) entries of $(AB)^T$ and $B^T A^T$)

using the properties 1 and 4:

$$B = A^T A \rightarrow B^T = (A^T A)^T = A^T A^{T^T} = A^T A = B$$

so matrix B is symmetric.

Question 3:

a)

one way to show that a matrix has a full rank or not is to calculate the determinant of that matrix. If the determinant is equal to 0 then the matrix does not have full rank. Otherwise, the rows and columns of the matrix are linearly independent and it has full rank. So in here:

$$M = \begin{bmatrix} 0.3 & 0.5 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 0.3 & 0.4 & 0.1 \end{bmatrix}$$

And the determinant of the matrix M is:

$$\det(M) = 0.3 * \det \begin{bmatrix} 0.1 & 0.5 \\ 0.4 & 0.1 \end{bmatrix} - 0.5 * \det \begin{bmatrix} 0.4 & 0.5 \\ 0.3 & 0.1 \end{bmatrix} + 0.4 * \det \begin{bmatrix} 0.4 & 0.1 \\ 0.3 & 0.4 \end{bmatrix}$$

$$\Rightarrow \det(M) = 0.3(0.01 - 0.2) - 0.5(0.04 - 0.15) + 0.4(0.16 - 0.03) = 0.053 \neq 0$$

So the matrix M has full rank. Now for the second matrix N, which is:

$$N = \begin{bmatrix} -0.7 & 0.5 & 0.4 \\ 0.4 & -0.9 & 0.5 \\ 0.3 & 0.4 & -0.9 \end{bmatrix}$$

Although it is possible to calculate the determinant to realize whether the matrix has full rank or not, We can already say that the matrix N does not have full rank, because the rows of this matrix are not linearly independent and:

$$First\ row + Second\ row = -Third\ row$$

b)

$$\begin{bmatrix} -0.7 & 0.5 & 0.4 \\ 0.4 & -0.9 & 0.5 \\ 0.3 & 0.4 & -0.9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the determinant of the matrix of the coefficients, A, is zero it is possible to obtain a nonzero response for the equation above.

using the second equation, we can write x_3 as a linear function of x_1 , x_2:

$$x_3 = -0.8x_1 + 1.8x_2$$

using equation 1 we have:

$$x_1 \approx 1.19x_2$$

any nonzero vector of x, which satisfies the above conditions is a valid response.

c)

Assume that we have a stochastic matrix A

$$A = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$$

So for every column of the matrix we have:

$$\sum_{i=1}^{n} x_{ij} = 1$$

Now we define a matrix B, which is equal to $A - I$.

$$B = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn} \end{bmatrix}$$

Now because B=A-I we can easily conclude that for every column of the matrix B:

$$\sum_{i=1}^{n} y_{ij} = 0$$

And this means that the rows of the matrix B are not linearly independent from each other and it is easy to see that for any k:

$$[x_{k1} \quad \cdots \quad x_{kn}] = -\sum_{i \neq k} [x_{i1} \quad \cdots \quad x_{in}]$$

So the matrix B does not have full rank.

We can get the exact same results for the matrix $C = I - A$.

Question 4:

a) We set

$$A_j = \sum_{i=1}^{30} a_{ji}$$

It is obvious that A_j is a real number and it shows the total amount of the endorsement that student j gets from his classmates.

b) Now we set

$$B_i = \sum_{i=1}^{30} a_{ij}$$

According to the question, B_i can have two possible values. If $B_i = 0$ it means that student i did not propose any endorsements for his classmates. On the other hand, if it is equal to one it means that student i made some endorsements for his classmates. Note that in this part endorsements from different students have the same weights.

c) It is easy to see that the matrix A is a stochastic matrix, and the sum of the elements of each column is equal to one (Note that we assumed here that all of the students should give endorsements about their classmates). So as we have seen before if A is a stochastic matrix then $A - I$ does not have full ranks.

d) Consider the case where we divide the class of 30 students into 15 groups and each group consists of only two students and these two only endorse each other and no one else. For example we can assume that student 1 only endorses student 2 and vice versa, and we have the same situation for students 3 and 4 till we get to students 29 and 30. Reminding the main equation:

$$Ax = x$$

We immediately realize that with the above conditions we will get the following results:

$$x_1 = x_2, x_3 = x_4, \ldots \quad , x_{29} = x_{30}$$

Moreover, we have the condition that $\sum_{i=1}^{30} x_i = 30$.

Consequently, we have 30 variables but we only have 16 equations to solve them, and this means that we have infinite solutions for these equations.

Question 5.

a)

$$Y(K,L) = A(\alpha K^{\rho} + (1-\alpha)L^{\rho})^{\frac{1}{\rho}}$$
$$\frac{d}{dK}Y(K,L) = A\alpha K^{\rho-1}(\alpha K^{\rho} + (1-\alpha)L^{\rho})^{\frac{1}{\rho}-1}$$
$$\frac{d}{dL}Y(K,L) = A(1-\alpha)L^{\rho-1}(\alpha K^{\rho} + (1-\alpha)L^{\rho})^{\frac{1}{\rho}-1}$$

Elasticity of substitution is defined as:

$$\varepsilon = \frac{\Delta\frac{k}{l}}{\Delta MRS}$$

where the denominator is:

$$\Delta MRS = \Delta\frac{dk}{dl} = \Delta\frac{\frac{dY}{dl}}{\frac{dY}{dk}}$$

It can be proved that for the CES function we have:

$$\varepsilon = \frac{1}{1-\rho}$$

b)

$$Y(K,L) = A(\alpha K^{\rho} + (1-\alpha)L^{\rho})^{\frac{1}{\rho}}$$
$$\lim_{\rho\to 0} Y(K,L) = ?$$

To solve this problem we first take logarithms from both sides and then use l'Hopital's rule. So

$$\log Y = \log A + \frac{\log(\alpha K^{\rho} + (1-\alpha)L^{\rho})}{\rho}$$
$$\lim_{\rho\to 0}\log Y(K,L) = \log A + \lim_{\rho\to 0}\frac{\log(\alpha K^{\rho} + (1-\alpha)L^{\rho})}{\rho}$$

using l'Hopital's rule, we have

$$\lim_{\rho \to 0} \log Y(K, L) = \log A + \lim_{\rho \to 0} \frac{(\alpha K^{\rho} \log K + (1 - \alpha) L^{\rho} \log L)}{\alpha K^{\rho} + (1 - \alpha) L^{\rho}}$$

$$= \log A + \alpha \log K + (1 - \alpha) \log L = \log(A K^{\alpha} L^{1-\alpha})$$

$$\Rightarrow \lim_{\rho \to 0} Y(K, L) = A K^{\alpha} L^{1-\alpha}$$