Mathematics for economists: ECON-C1000
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Supplementary readings on logic, sets and functions

1 Logic

1.1 Propositions

Definition 1 *A proposition is a statement which can be declared to be true or false without ambiguity.*

We say that "true"and "false" are the two possible truth values of a proposition.

Example 2 *"$2+2=4$", "Economics students do not need to understand mathematics", "$x<2$" is not a proposition since we cannot declare the truth value without knowing the value of x. It is a propositional function.*

Definition 3 *A propositional or (sentential) function is a statement which becomes a proposition when we replace the variable with a specified value.*

We denote propositions with capital letters, like A, and propositional functions indicating the variable, $A(x)$.

Definition 4 *A connective is a symbol which allows us to construct a new proposition $\mathcal{A}$ from simple propositions A,B,C,...,where $\mathcal{A}$ is called a compund proposition.*

A truth table shows the possible truth values of a compound proposition for all possible truth values of simple proposition. The basic connectives are Negation ($\sim$ or $\neg$),

A	$\neg$ **A**
T	F
F	T

Conjunction ("and") ($\wedge$),

A	**B**	**A** $\wedge$ **B**
T	T	T
T	F	F
F	T	F
F	F	F

and Disjunction ("inclusive or") ($\vee$),

A	**B**	**A** $\vee$ **B**
T	T	T
T	F	T
F	T	T
F	F	F

Note that $A \vee B$ is T if A or B or both are true. Alternatively we can define the "exclusive or" where the compound proposition is true if either A or B is true, but not both. The Implication ("if....then...) ($\Rightarrow$) is represented by the truth table

A	**B**	**A** $\Rightarrow$ **B**
T	T	T
T	F	F
F	T	T
F	F	T

In $A \Rightarrow B$, A is called the "hypothesis"and B the "conclusion". We say that "A is a sufficient condition for B" and "B is a necessary condition for A". The connective Equivalence ("if and only if") ($\Leftrightarrow$) is represented by the truth table,

A	**B**	**A** $\Leftrightarrow$ **B**
T	T	T
T	F	F
F	T	F
F	F	T

An equivalence is hence true if both propositions have the same truth value.

Definition 5 *A compound proposition is a tautology if it is always true and a contradiction if it is always false.*

Show that "$A \wedge (\neg A)$", "$A \Rightarrow A$", "$A \Rightarrow (A \vee B)$" are tautologies, while "$A \vee (\neg A)$" is a contradiction.

Definition 6 *Two compound propositions A and B are logically equivalent if $A \Leftrightarrow B$ is a tautology.*

Two logical equivalents will be used commonly: "$A \Rightarrow B$" and "$\neg B \Rightarrow \neg A$" (contraposition)as well as "$A \Leftrightarrow B$" and "$A \Rightarrow B \wedge B \Rightarrow A$". Note that starting from an implication "$A \Rightarrow B$" we can construct the contrapositive:" $\neg B \Rightarrow \neg A$" which is equivalent to the original implication and the converse "$B \Rightarrow A$", whose truth value does not depend on the original implication.

1.2 Quantifiers

The universal quantifier ("for all") is denoted by $\forall$. By $\forall x,\ A(x)$ we mean that $A(x)$ is true for all x. The existential quantifier ("there exists" or "for some") is denoted by $\exists$. By $\exists x,\ A(x)$ we mean that there exists an x for which $A(x)$ is true. The numerical quantifier "there exists a unique" is denoted by $\exists!$ and $\exists x!,\ A(x)$ claims that there exists a unique x for which $A(x)$ is true. Finally $\neg\exists$ means "there exists no".

Example 7 *"$\forall x \in \mathbb{R},\ x > 2$", $\exists x \in \mathbb{R},\ x > 2$.*

Negating quantified propositions works as follows. If the original proposition is $\forall x,\ A(x)$, its negation is $\neg(\forall x,\ A(x))$ (equivalent to $\exists x,\ \neg A(x)$). Special care should be taken when dealing with propositions that contain multiple quantifiers. In particular, the proposition $\forall x \exists y,\ A(x,y)$ meaning "for all x there exists a y such that $A(x,y)$ is true" is different from $\exists x \forall y$, $A(x,y)$ which means that "there exists an x such that for all y $A(x,y)$ is true".

1.3 Methods of Proof

A proof consists of a sequence of propositions one implied by the other,

$$\underbrace{H}_{\text{Hypothesis}} \Rightarrow \underbrace{A_1 \Rightarrow \cdots \Rightarrow A_n}_{\text{Intermediate steps}} \Rightarrow \underbrace{C}_{\text{Conclusion}}$$

1. **Direct Proof.** Show that $H \Rightarrow C$.

2. **Proof by contrapositive.** We know that $H \Rightarrow C$ is logically equivalent to $(\neg C) \Rightarrow (\neg H)$. So assuming that $(\neg C)$ is true we show that $(\neg H)$ is true.

3. **Proof by contradiction.** Assume $\neg (H \Rightarrow C)$. Then show that $[\neg (H \Rightarrow C) \wedge H]$ is a contradiction. Actually it is enough to show that $[(\neg C) \wedge H]$ is a contradiction because $[\neg (H \Rightarrow C) \wedge H]$ and $[(\neg C) \wedge H]$ are logically equivalent.

4. **Proof by construction.** For example, if the conclusion is in the form "$\exists x,\ A(x)$" find a value x such that $A(x)$ is true. Note also that proof by construction is useful in disproving proposition i.e. in finding counterexamples to candidate propositions.

5. **Proof by decomposition.** $H \Rightarrow [(A_1 \vee A_2) \wedge \neg (A_1 \vee A_2)]$, then show that $A_1 \Rightarrow C$ and $A_2 \Rightarrow C$.

Examples for the different proof techniques are given below:

1. Direct. $H : x > 0,\ C : 2x > 0$.

   $x > 0 \Rightarrow 2x > 0$ (multiply by 2).

2. Contrapositive. $H : x > 0,\ c : 2x > 0$

   $2x \leq 0 \Rightarrow x \leq 0$ $\left(\text{multiply by } \frac{1}{2}\right)$

3. Contradiction. $H : x > 0,\ c : 2x > 0$

   $2x \leq 0 \wedge x > 0 \Rightarrow x \leq 0 \wedge x > 0$ and this is false.

4. Construction. $H : x \in \mathbb{R},\ C : \exists x,\ x > 2$

   For $x = 10$, we have $x > 2, x \in \mathbb{R}$.

5. Decomposition. $H : x \in \mathbb{R},\ C : |x| \geq 0$;

   Either $x \geq 0$, or $x < 0$. and then show that $x \geq 0 \Rightarrow |x| \geq 0$ and $x < 0 \Rightarrow |x| \geq 0$.

A famous example of proof by contradiction is the following.

Example 8 *Hypothesis:* $x^2 = 2$. *Conclusion:* x *is not a rational number.*
Proof. *Suppose that the hypothesis is true and the conclusion is not true. In other words,* $x^2 = 2$ *and* $x = \frac{m'}{n'}$ *for some nonzero integers* m', n'. *We need to find an equivalent for of this negation of the conclusion. By writing* m' *and* n' *as products of prime numbers and cancelling common terms in the numerator and denominator, we can write* $x = \frac{m}{n}$ *for nonzero integers* m *and* n *such that they have no common factors. Hence we want to show that* m *and* n *having no common factors and* $\left(\frac{m}{n}\right)^2 = 2$ *is a contradiction.*

By hypothesis:

$$x^2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2} = 2.$$

Multiply both sides of the last equation by n^2 *(recall that* n *is nonzero) to obtain*

$$m^2 = 2n^2.$$

Hence m^2 *is an even number. As a number is even if and only if its factorization into prime numbers contains 2, this implies that* m *itself must be even. But then there is a* p *such that*

$$m = 2p.$$

Thus

$$m^2 = 2n^2 \Leftrightarrow (2p)^2 = 2n^2 \Leftrightarrow 4p^2 = 2n^2 \Leftrightarrow 2p^2 = n^2.$$

But repeating the previous steps in the argument, this leads to the conclusion that there is a q *such that*

$$n = 2q.$$

But then m *and* n *have a common factor and we have reached a contradiction and the original claim is thus proved.* ■

1.4 Exercises

1. Construct the truth tables for the following propositions. Which are tautologies and which are contradictions?

   (a) $[A \wedge B] \Rightarrow [(A \vee B) \Leftrightarrow B]$,

   (b) $[(A \Rightarrow B) \wedge (B \Rightarrow C)] \Rightarrow (A \Rightarrow C)$,

   (c) $(A \Rightarrow B) \Leftrightarrow (\neg B \Rightarrow \neg A)$,

(d) $(A \Rightarrow B) \Leftrightarrow (B \Rightarrow A)$.

2. Argue that $\forall x \forall y,\, A(x,y) \Leftrightarrow \forall y \forall x\, A(x,y)$ and $\exists x \exists y,\, A(x,y) \Leftrightarrow \exists y \exists x,\, A(x,y)$ and find a statement $A(x,y)$ such that $\forall x \exists y,\, A(x,y)$ is true but $\exists x \forall y,\, A(x,y)$ is false.

3. Find the negation of the following propositions: (all the variables here are taken to be real numbers).

   a) $\forall x \geq 1,\ x^3 \geq x^2$.

   b) $\forall x \geq 0 \forall a \geq 0,\ ax \geq 0$.

   c) $\forall x \exists y,\ |x-y| < \varepsilon \Rightarrow |x^2 - y^2| < \varepsilon^2$.

   d) (harder) $\forall \varepsilon > 0 \exists \delta > 0 \forall x \forall y,\ |x-y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon$.

4. An irrational number x is called tricky if there is an irrational number y such that x^y is rational. Give a proof by decomposition to show that tricky numbers exist. (Hint: Consider $\sqrt{2}^{\sqrt{2}}$.)

2 Sets

A set is a collection of objects which we call its elements. When an object x is an element of a set X, one writes $x \in X$. Otherwise, one writes $x \notin X$. A set is called a *singleton* set if it contains exactly one element. The set containing no object is called *the empty set,* and is denoted ϕ.The set to which all the objects belong we wish to consider is called the *universal set* Ω.

If A and B are sets, then one writes $A = B$ if they contain exactly the same objects, i.e., if $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$. Otherwise, one writes $A \neq B$. If only the first inclusion holds, A is called a *subset* of B and one writes $A \subset B$. In particular, every set is a subset of itself (i.e.,equality is permitted). If A is a subset of B and $A \neq B$, then A is called a *proper* subset of B.

Note that the empty set ϕ is a subset of *every* set A (including itself). A set is called *finite* if it has finitely many elements, otherwise it is called infinite. In particular, ϕ is finite. For any finite set A, $\#A$ will denote its number of elements. We will sometimes write $\#A = +\infty$ if a set A has infinitely many elements. Some useful (infinite) number sets:

$\mathbb{R}$= $\{x : x$ is a real number$\}$,

$\mathbb{R}_+ = \{x : x$ is a nonnegative real number$\} = \{x \in \mathbb{R} : x \geq 0\}$,

$\mathbb{R}_{++} = \{x :$ is a positive real number$\} = \{x \in \mathbb{R} : x > 0\}$,

$\mathbf{Z} = \{x : x$ is an integer$\} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$,

$\mathbf{N} = \{x : x$ is a natural number$\} = \{x : x$ is a positive integer$\} = \{x \in \mathbf{Z} : x > 0\}$,

$\mathbf{Q} = \{x : x$ is a rational number$\} = \{x : x = m/n$ for some $m \in \mathbf{Z}$ and $n \in \mathbf{N}\}$.

Definition 9 *If A and B are sets, then the complement of B relative to A is the set of all elements of A which do not belong to B. We denote this set by $A \backslash B = \{x \in A : x \notin B\}$, related notations are $A - B$, or $A \sim B$.*

Definition 10 *If A is a set, then the set of all subsets is called the power set of A, denoted by $\mathcal{P}(A)$ or 2^A.*

It is called the "power set" since if A has n elements, $\mathcal{P}(A)$ has 2^N elements.

2.1 Set Operations

For any pair of real numbers a and b such that $a < b$, we write $[a, b]$ and (a, b) for the two intervals $\{x \in \mathbb{R} : a \leq x \leq b\}$and $\{y \in \mathbb{R} : a < y < b\}$, respectively. Such intervals are called *bounded* while, for example, the interval $\mathbb{R}_+ = [0, +\infty)$ is called *unbounded*. Intervals of the first type, i.e. those containing both end points, are called *closed* while intervals of the second type, i.e., those which contain none of their two end-points, are called *open*. Intervals of the "intermediate" type $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ are neither open nor closed.

While inclusion is a (partial binary) *relation* between sets, intersection and union formation is an *operation* on sets, producing new sets from any given collection of sets. First, suppose A and B are sets. Then

$$A \cap B = \{x : x \in A \text{ and } x \in B\} \text{ and}$$
$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both.}\}.$$

Note in particular, that for any set A, $A \cap A = A \cup A = A, A \cap \phi = \phi$ and $A \cup \phi = A$. Two sets are said to be *disjoint* if their intersection is (the) empty (set).

These familiar operations can be generalized from pairs of sets to arbitrarily large collections of sets. Let I be an arbitrary nonempty set (not necessarily a number set), to be called an *index* set, and for each $i \in I$, let A_i be a set. Then

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for all } i \in I\} \text{ and } \bigcup_{i \in I} A_i = \{y : y \in A_i \text{ for some } i \in I\}.$$

Example 11

$$\bigcap_{n \in N} [n-1, n] = \phi, \bigcup_{n \in N} [n-1, n] = \mathbb{R}_+ \text{ and } \bigcup_{n \in N} \left[\frac{1}{n+1}, \frac{n}{n+1}\right] = (0, 1).$$

The difference of set A and B in A is defined as follows:

$$A \setminus B = \{x \in A : x \notin B\}.$$

The complement of a set A in the universal set Ω is given by

$$A^C = \Omega \setminus A = \{x \in \Omega : x \notin A\}.$$

Definition 12 *If A and B are two non-void sets, then the Cartesian product $A \times B$ of A and B is the set of all ordered pairs (a, b) with $a \in A$, and $b \in B$.*

The Cartesian product can be extended to a finite or infinite collection of sets

$$\underset{i=1}{\overset{N}{\times}} A_i = \{(x_1, ..., x_n), \ x_i \in A_i, \ i = 1, ..., n\}.$$

2.2 Exercises

1. Show that the set D of all elements that belong either to A or to B but not to both is given by:

   $$D = (A \setminus B) \cup (B \setminus A).$$

   The set D is often called the **symmetric difference** of A and B and denoted by $A \triangle B$. Represent it by a Venn diagram. Show that the symmetric difference D is also given by

   $$D = (A \cup B) \setminus (A \cap B).$$

2. (De Morgan's laws) Prove the following:

   a) $(A \cup B)^C = A^C \cap B^C$.

   b) $(A \cap B)^C = A^C \cup B^C$.

   c) $A \subset B \Leftrightarrow B^C \subset A^C$.

3 Functions

Definition 13 *Consider two sets A and B. Suppose that with each element of $x \in A$ there is associated an element of B, which we denote by $f(x)$. Then f is said to be a function or a mapping from A to B and we write*

$$f : A \to B.$$

The set A is called the domain of and the elements $f(x)$ are called the values of f. The set of all values of f is called the range of f.

Definition 14 *Let A and B be two sets and let f be a function from A into B. If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$ for $x \in E$. We call $f(E)$ the image of E under f and $f(A)$ is the range of f.*

Definition 15 *If $f(A) = B$, then f maps A onto or f is surjective.*

Definition 16 *If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the inverse image of E under f^{-1}. If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$, such that $f(x) = y$.*

Definition 17 *If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A, then f is said to be a one-to-one or injective mapping of A into B.*

Note that we could also say that f is one-to-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Definition 18 *If there exists a 1-1 mapping f of A onto B, we say that A and B can be put in 1-1 correspondence, and f is said to be bijective.*

Observe that if a function f is bijective, then the inverse function $f^{-1} : B \to A$ is well defined and $a = f^{-1}(b)$ if and only if $b = f(a)$.

Suppose next that we have two functions, $f : A \to B$ and $g : B \to C$. We can define a composite function $h : A \to C$ denoted by $g \circ f$ as follows.

Definition 19 *Let $f : A \to B$ and $g : B \to C$. Then the composite function $h = g \circ f : A \to C$ is defined by $h(a) = g(f(a))$.*

A composite function h is bijective if and only if both f and g are bijective. Notice also that the composite function of $f : A \to B$ and $f^{-1} : B \to A$ is given by the identity function $i : A \to A$ that satisfied $i(a) = a$ for all $a \in A$. In other words, for all $a \in A$, $a = f^{-1}(f(a))$ and for all $b \in B$, $b = f(^{-1}(a))$.

3.1 Exercises

1. Let $f : S \to T$ be a function. If $Y \subseteq T$, we denote by $f^{-1}(Y)$ the largest subsets of S which f maps into Y. That is

$$f^{-1}(Y) = \{x : x \in S \text{ and } f(x) \in Y\}.$$

   The set $f^{-1}(Y)$ is called the inverse image of Y under f. Prove the following for arbitrary subsets X of S and Y of T.

   (i) $X \subseteq f^{-1}[f(X)]$,

   (ii) $f[f^{-1}(Y)] \subseteq Y$,

   (iii) $f^{-1}[Y_1 \cup Y_2] = f^{-1}(Y_1) \cup f^{-1}(Y_2)$,

   (iv) $f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$.

2. Let $f : S \to T$ be a function. Prove that the following statements are equivalent.

   (a) f is one-to-one on S,

   (b) $f(A \cap B) = f(A) \cap f(B)$ for all subsets A,B of S,

   (c) $f^{-1}[f(A)] = A$ for every subset A of S.

   (d) For all disjoint subsets A and B of S, the images $f(A)$ and $f(B)$ are disjoint.

   (e) For all subsets A and B of S with $B \subseteq A$, we have

$$f(A - B) = f(A) - f(B).$$

   (Hint: We may define

$$f(\emptyset) = \emptyset.)$$

4 The Structure of $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbb{R}$

4.1 Countability

Definition 20 *Two sets are called similar or equinumerous if there exists a bijective function $f : A \to B$.*

We write $A \sim B$ and say that f establishes a one-to-one correspondence between A and B.

Definition 21 *For any positive integer n, let J_n be the set whose elements are the integers $1, 2, ..., n$;let J be the set consisting of all positive integers. For any set A we say:*
(a) A is finite if $A \sim J_n$ for some n,
(b) A is infinite if A is not finite,
(c) A is countable if $A \sim J$,
(d) A is uncountable if A is neither finite nor countable.

The integer n is called the cardinal number of A.

Theorem 22 *Every subset of a coutable set is countable.*

Example 23 *Let $\mathbf{Z} = \{0, 1, -1, 2, -2, \}$ the set of integers. Then $\mathbf{N} \sim \mathbf{Z}$. Finding a $1-1$ correspondence between $\mathbf{N}$ and $\mathbf{Z}$ is left as an easy exercise.*

Theorem 24 *If F is a countable collection of countable sets, then the union of all sets in F is also a countable set.*

Example 25 *Consider $\mathbf{Q}$. Denote by A_n the set of all positive rational numbers with denominator $n \in \mathbf{N} \backslash \{0\}$. The set of all positive rational numbers is equal to $\bigcup_{i=1}^{\infty} A_i$. By the previous theorem then, the set of rational numbers is countable.*

4.2 Mathematical Induction

Definition 26 *A set S of numbers is said to be inductive if and only if*
(a) $1 \in S$,
(b) $(x + 1) \in S$ whenever $x \in S$.

Definition 27 *A real number n is said to be a natural number if and only if it belongs to every inductive set of real numbers. The set of all natural numbers will be denoted by* $\mathbf{N}$.

Theorem 28 *The set* $\mathbf{N}$ *of all natural numbers is an inductive set.*

Theorem 29 ***Principle of mathematical induction.*** *If S is an inductive set of natural numbers, then* $S = \mathbf{N}$.

Example. Show that

$$1 + 2 + \cdots + n = \frac{n\,(n+1)}{2}$$

for every natural number n. Solution. Let S be the set of natural numbers n for which the formula holds. we shall show that S is an inductive set. Clearly $1 \in S$. Suppose $k \in S$ and the formula holds with $k = n$. Adding $(k+1)$ to both sides we se that

$$1 + 2 + \cdots + k + (k+1) = \frac{k\,(k+1)}{2} + (k+1) = \frac{(k+1)\,(k+2)}{2},$$

which is the formula for $n = k+1$, which concludes that $(k+1) \in S$.

4.3 Exercises

1. Show by induction that $n! > 2^n$ for each natural number $n > 3$.
2. For $n \in \mathbf{N}$, we define $n! = 1 \cdot 2 \cdot ... \cdot n$. We also define $0! = 1$. Then

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!\,(n-k)!}$$

where $n \geq k$, $n \in \mathbf{N}$, $k \in \mathbf{N}$, and $\left(\begin{array}{c} n \\ k \end{array} \right)$represents the number of different combinations of n objects in non-ordered k-tuples. Prove the binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) a^{n-k} b^k.$$

Hint: Prove first that

$$\left(\begin{array}{c} n \\ k \end{array} \right) + \left(\begin{array}{c} n \\ k-1 \end{array} \right) = \left(\begin{array}{c} n+1 \\ k \end{array} \right).$$