Mathematics for Economists
ECON-C1000
Spring 2021
Juuso Välimäki

Problem Set 3: Due May 13, 2021

1. Consider the following matrices.

   (a) Determine if the matrices are positive or negative definite:

   $$\text{i)} \begin{pmatrix} 1 & 3 \\ 3 & 5 \end{pmatrix} \qquad \text{ii)} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad \text{ii)} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{pmatrix}$$

   (b) Show that all matrices of the form $X^\top X$ (where X is an arbitrary $k \times n$ matrix) are positive definite.

   (c) Find the critical points of the following function:

   $$f(x, y) = \frac{1}{2}(x^2 + 2bxy + 9y^2),$$

   and determine the values of b such that the critical points are minima.

2. Determine the nature of the critical points:

   (a) $f(x, y) = -1 + 4(e^x - x) - 5x \sin y + 6y^2$ at $x = y = 0$.

   (b) $f(x, y) = (x^2 - x) \cos y$ at $x = 1, y = \pi$.

3. Consider the function:

   $$f(x) = \frac{x^2}{x - 3}$$

   when $-7 < x < 3$.

(a) Find any critical points $\hat{x}$, i.e. points where $f'(\hat{x}) = 0$.

(b) Find the second order Taylor approximation to the function around $\hat{x}$

$$f(\hat{x} + h) = f(\hat{x}) + f'(\hat{x})h + \frac{1}{2}f''(\hat{x})h^2$$

and determine if f has a local minimum or maximum at $\hat{x}$.

4. Consider the function $f(x_1, x_2, x_3) = 10x_1^{\frac{1}{3}}x_2^{\frac{1}{2}}x_3^{\frac{1}{6}}$.

   (a) Approximate the value of this function around the point $x = (27, 16, 64)$ using the differentials when x_1 increases to 27.1, x_2 decreases to 15.7, and x_3 remains the same.

   (b) Compare the approximated value you obtained in the previous answer with the actual value of the function at the new point.

   (c) Answer again to questions 1. and 2. for $dx_1 = dx_2 = 0.2$ and $dx_3 = -0.4$.

5. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable utility function, and let $f : \mathbb{R} \to \mathbb{R}$ be differentiable function with a strictly positive derivative for every $x \in \mathbb{R}$. Define the composite function $v(x, y) := f(u(x, y))$. Recall that the Marginal Rate of Substitution of u at a point (x_0, y_0) is

$$MRS_{x,y} = \frac{\frac{\partial u(x_0,y_0)}{\partial x}}{\frac{\partial u(x_0,y_0)}{\partial y}}.$$

   (a) Write the expression of the MRS at (x_0, y_0) for the composite function v.

   (b) Use the chain rule to show that the MRS of u and v at (x_0, y_0) is the same.

   (c) Now assume that u is also homogeneous of degree k. Show that the MRS of u is a homogeneous function of degree zero.

6. (Implicit function theorem without specified functional forms) A competitive firm maximizes its profit by choosing optimally its inputs:

$$\max_{k,l \geq 0} pf(k,l) - wl - rk,$$

where $f(k,l)$ is the production function, k is capital, l is labor, r is the rental cost of capital, w the market wage for labor and p is the output price.

(a) What are the natural endogenous variables for this model? What are the exogenous variables?

(b) Write the first-order conditions for optimal choices of the endogenous variables.

(c) When can you use the implicit function theorem to determine the changes in the endogenous variables for small changes in the exogenous variables?

(d) Compute the sign of changes in the endogenous variables when the exogenous variables change.