Mathematics for Economists

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Solutions to the problem set 3:

Question 1:

a)

i)

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 5 \end{bmatrix}$$

$$M_1 = \det(1) = 1$$

$$M_2 = \det(A) = -4$$

so the matrix A is indefinite.

ii)

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$M_1 = \det(1) = 1$$

$$M_2 = \det(B) = 0$$

so the matrix B is positive semi definite.

iii)

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 9 \end{bmatrix}$$

$$M_1 = \det(1) = 1$$

$$M_2 = \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\right) = 1$$

$$M_3 = \det(C) = 1(45 - 16) - 2(18 - 12) + 3(8 - 15) = 29 - 12 - 21 = -4$$

so matrix C is indefinite.

b)

According to the definition, matrix A is positive semi-definite if for any x, $x^t Ax \geq 0$.

Now assuming that $A = X^T X$ where $X_{k \times n}$ and z is any arbitrary vector with length n, we have:

$$z^T(X^T X)z = (Xz)^T(Xz) = \|Xz\|_2^2 \geq 0$$

Note that Xz is a vector of $k \times 1$.

so $X^T X$ is a positive semi-definite matrix .

If we add a constraint "X has rank k" we can prove that $X^T X$ is POSITIVE definite, because then we know there is no non-zero z for which Xz=0.

c)

$$f(x,y) = \frac{1}{2}(x^2 + 2bxy + 9y^2)$$

to find the critical point of the function f, we should set the gradient equal to zero, so:

$$\frac{df}{dx} = x + by = 0$$

$$\frac{df}{dy} = 9y + bx = 0$$

the critical point is $(x_0 = 0, y_0 = 0)$. Now we determine the value of b so that the matrix A is positive definite and the critical point is local minimum.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{b}{2} \\ \frac{b}{2} & \frac{9}{2} \end{bmatrix}$$

the matrix A should be positive definite so $\det(A) > 0$

$$\frac{9}{4} - \frac{b^2}{4} > 0 \to 9 - b^2 > 0 \to -3 < b < 3$$

Question 2:

a)

$$f(x,y) = -1 + 4(e^x - x) - 5xsin(y) + 6y^2$$

$$\frac{df}{dx} = 4e^x - 4 - 5\sin(y)$$

$$\frac{df}{dy} = -5xcos(y) + 12y$$

$$Hf = \begin{bmatrix} \frac{d^2f}{dxdx} & \frac{d^2f}{dxdy} \\ \frac{d^2f}{dydx} & \frac{d^2f}{dydy} \end{bmatrix} = \begin{bmatrix} 4e^x & -5\cos(y) \\ -5\cos(y) & 5xsin(y) + 12 \end{bmatrix}$$

at $(x, y) = (0,0)$ we have:

$$Hf = \begin{bmatrix} 4 & -5 \\ -5 & 12 \end{bmatrix}$$

which is a positive definite matrix since $\det(Hf) = 48 - 25 = 23 > 0$

so the point $(x, y) = (0,0)$ is the minima of the function f.

b)

$$f(x, y) = (x^2 - x)\cos(y) \ \ and \ (x_0, y_0) = (1, \pi/2)$$

$$\frac{df}{dx} = (2x - 1)\cos(y)$$

$$\frac{df}{dy} = (x^2 - x)(-\sin(y))$$

$$Hf = \begin{bmatrix} \frac{d^2f}{dxdx} & \frac{d^2f}{dxdy} \\ \frac{d^2f}{dydx} & \frac{d^2f}{dydy} \end{bmatrix} = \begin{bmatrix} 2\cos(y) & (2x-1)(-\sin(y)) \\ (2x-1)(-\sin(y)) & (x^2 - x)(-\cos(y)) \end{bmatrix}$$

at $(x, y) = (1, \pi/2)$ we have:

$$Hf = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

whish is indefinite, so the point $(x, y) = (0,0)$ is not a minima nor maxima of the function f.

Question 3:

$$f(x) = \frac{x^2}{x - 3}, -7 < x < 3$$

$$f'(x) = \frac{2x(x-3) - x^2}{(x-3)^2}$$

$$f'(x) = 0 \rightarrow x(x - 6) = 0 \rightarrow x = 0, x = 6$$

but we know that $-7 < x < 3$, so $\hat{x} = 0$ is the only critical point of the function f.

To derive the second order taylor approximation, we also need $f''(0)$, so

$$f''(x) = \frac{2(x-3)^2 - 2(x^2 - 6x)}{(x-3)^3}$$

$$f''(0) = \frac{-2 * 9}{27} = -\frac{2}{3} < 0$$

so the critical point $\hat{x} = 0$ is the local maximum of the function f.

the taylor approximation:

$$f(\hat{x}+h) = f(\hat{x}) + f'(\hat{x})h + \frac{1}{2}f''(\hat{x})h^2 = 0 + 0 - \frac{1}{2}\frac{2}{3}h^2 = -\frac{h^2}{3}$$

Question 4:

$$f(x,y,z) = 10x^{\frac{1}{3}}y^{\frac{1}{2}}z^{\frac{1}{6}}\,, (a,b,c) = (27\,,16\,,64)$$

a)

using taylor approximation for multi-variable functions, we have:

$$f(x,y,z) = f(a+da\,, b+db\,, c+dc) =$$

$$= f(a,b,c) + f'_x(a,b,c)da + f'_y(a,b,c)db + f'_z(a,b,c)dc$$

$$f'_x = \frac{df}{dx} = \frac{10}{3}x^{-\frac{2}{3}}y^{\frac{1}{2}}z^{\frac{1}{6}}$$

$$f'_y = \frac{df}{dy} = 5x^{\frac{1}{3}}y^{-\frac{1}{2}}z^{\frac{1}{6}}$$

$$f'_z = \frac{df}{dz} = \frac{5}{3}x^{\frac{1}{3}}y^{\frac{1}{2}}z^{-\frac{5}{6}}$$

using the point (a,b,c) we have:

$$f(27,16,64) = 240$$

$$f'_x = \frac{80}{27}\,, f'_y = \frac{15}{2}\,, f'_z = \frac{5}{8}\ and$$

$$da = 0.1\,, db = -0.3\,, dc = 0$$

finally:

$$f(27.1\,,15.7\,,64) \approx 240 + \frac{80}{27}*0.1 - \frac{15}{2}*0.3 + 0 = 238.04$$

b)

the actual value of the function is equal to:

$$f(27.1\,,15.7\,,64) = 237.77$$

c)

$$f(27.2\,,16.2\,,63.6) \approx 240 + \frac{80}{27}*0.2 + \frac{15}{2}*0.2 - \frac{5}{8}*0.4 = 241.84$$

and the actual value is

$$f(27.2\,,16.2\,,63.6) = 241.57$$

Question 5:

a)

composite function $v(x,y) = f(u(x,y))$

$$MRS_{u(x,y)} = \frac{\frac{\partial u(x_0,y_0)}{\partial x}}{\frac{\partial u(x_0,y_0)}{\partial y}}$$

a,b)

$$MRS_{v(x,y)} = \frac{\frac{\partial v(x_0,y_0)}{\partial x}}{\frac{\partial v(x_0,y_0)}{\partial y}} = \frac{\frac{\partial f(u(x_0,y_0))}{\partial u} . \frac{\partial u(x_0,y_0)}{\partial x}}{\frac{\partial f(u(x_0,y_0))}{\partial u} . \frac{\partial u(x_0,y_0)}{\partial y}} = \frac{\frac{\partial u(x_0,y_0)}{\partial x}}{\frac{\partial u(x_0,y_0)}{\partial y}} = MRS_{u(x,y)}$$

c)

Definition of the homogenous function:

A real valued function $f(x_1, x_2, \dots, x_n)$ is homogenous of degree k if for all $t > 0$:

$$f(tx_1, tx_2, \dots, tx_n) = t^k f(x_1, x_2, \dots, x_n)$$

so considering u as a homogenous function of degree k, we have:

$$MRS_{u(tx,ty)} = \frac{\frac{\partial u(tx_0,ty_0)}{\partial x}}{\frac{\partial u(tx_0,ty_0)}{\partial y}} = \frac{\frac{t^k \partial u(x_0,y_0)}{\partial x}}{\frac{t^k \partial u(x_0,y_0)}{\partial y}} = \frac{\frac{\partial u(x_0,y_0)}{\partial x}}{\frac{\partial u(x_0,y_0)}{\partial y}} = MRS_{u(x,y)}$$

Question 6:

a)

We assume that prices are given from the outside of the firm, so p, w and r are exogenous variables. On the other hand, it is possible for us to calculate the exact amount of the capital and labour to increase out profit so l and k are endogenous variables.

b)

our endogenous variables are k and l, so:

$$\frac{\partial f(k,l)}{\partial k} - \frac{r}{p} = 0$$

$$\frac{\partial f(k,l)}{\partial l} - \frac{w}{p} = 0$$

c)

We rewrite the problem as follows:

$$f_1(k,l;p,r,w) = g(k,l) - \frac{r}{p}$$
$$f_2(k,l;p,r,w) = h(k,l) - \frac{w}{p}$$

where $g(k,l) = \frac{\partial f(k,l)}{\partial k}$ and $h(k,l) = \frac{\partial f(k,l)}{\partial l}$. We should assume that the system of the equations are satisfied at $(\bar{k},\bar{l},\bar{p},\bar{r},\bar{w})$, so in the next step we make the matrices of partial derivative (we assume y as the endogenous and x as the exogenous variables):

$$D_y f(\bar{k},\bar{l};\bar{p},\bar{r},\bar{w}) = \begin{bmatrix} \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial l} \\ \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial l} \end{bmatrix} = \begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix}$$

And

$$D_x f(\bar{k},\bar{l};\bar{p},\bar{r},\bar{w}) = \begin{bmatrix} \frac{\partial f_1}{\partial p} & \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial p} & \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{r}{p^2} & -\frac{1}{p} & 0 \\ \frac{w}{p^2} & 0 & -\frac{1}{p} \end{bmatrix}$$

Now assuming the fact that functions f and g are continuously differentiable at $(\bar{k},\bar{l},\bar{p},\bar{r},\bar{w})$, the necessary condition is:

$$\det\left(D_y f(\bar{k},\bar{l};\bar{p},\bar{r},\bar{w})\right) \neq 0$$

so:

$$\frac{\partial g(\bar{k},\bar{l})}{\partial k} \cdot \frac{\partial h(\bar{k},\bar{l})}{\partial l} \neq \frac{\partial g(\bar{k},\bar{l})}{\partial l} \frac{\partial h(\bar{k},\bar{l})}{\partial k}$$

d)

Using implicit function theorem we have:

$$D_y f(\hat{y},\hat{x})dy + D_x f(\hat{y},\hat{x})dx = 0$$

So

$$\begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix} \begin{bmatrix} dk \\ dl \end{bmatrix} + \begin{bmatrix} \frac{r}{p^2} & -\frac{1}{p} & 0 \\ \frac{w}{p^2} & 0 & -\frac{1}{p} \end{bmatrix} \begin{bmatrix} dp \\ dr \\ dw \end{bmatrix} = 0$$

Using Cramer's rule:

$$dk = \frac{det\begin{bmatrix}\frac{1}{\rho} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ 0 & \frac{\partial h(\bar{k},\bar{l})}{\partial l}\end{bmatrix}dr}{\det\begin{bmatrix}\frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l}\end{bmatrix}}$$

We assume that the denominator of the above formula is always positive. so

$$\frac{\partial g(\bar{k},\bar{l})}{\partial k}.\frac{\partial h(\bar{k},\bar{l})}{\partial l} > \frac{\partial g(\bar{k},\bar{l})}{\partial l}\frac{\partial h(\bar{k},\bar{l})}{\partial k}$$

why? It simply means that the cross effects of k and l are not too strong over each other. In other words, adding Labor does not change the marginal product of k (MP_k) too much and vice versa.

considering the previous assumption, we easily conclude that the sign of $\frac{dk}{dr}$ is the same as the sign of

$$\frac{1}{\rho}\frac{\partial h(\bar{k},\bar{l})}{\partial l}$$

We can compute $\frac{dl}{dr}$ by doing the same:

$$dl = \frac{det\begin{bmatrix}\frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{1}{\rho} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & 0\end{bmatrix}dr}{\det\begin{bmatrix}\frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l}\end{bmatrix}}$$

And the sign of $\frac{dl}{dr}$ is the same as the sign of

$$-\frac{1}{\rho}\frac{\partial h(\bar{k},\bar{l})}{\partial k}$$