Name________________________ Student number________________

Home assignment 2

On grid $i \in \{0,1,2,3\}$, particle surrogate method (PSM) gives the second order ordinary differential equations

$$\frac{S}{h}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + \rho A h \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix} = 0$$

for a vibration problem of a string of length L. Assuming that the horizontal tightening S, cross-sectional area A, density of material ρ, and spacing h of the grid points are constants, derive the angular speeds and the corresponding modes of the free vibrations.

Solution

Calculation of the angular velocity-mode pairs $(\omega, \mathbf{A})$ is based on the use of trial solution $\mathbf{a}(t) = \mathbf{A}\exp(\mathrm{i}\omega t)$ which transform the set of ordinary differential equations into algebraic ones. In the present case and

$$\frac{S}{h}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + \rho A h \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix} = 0 \;\Rightarrow\; (\frac{S}{h}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \rho A h \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 .$$

where $h = L/3$. To simplify the calculations, let us write the equations first with notation $\lambda = \omega^2 \rho A h^2 / S$

$$(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 .$$

The homogeneous linear equation system can yield a non-zero solution to the mode only if the matrix in parenthesis is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

$$\det\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = 0 \quad \text{so} \quad \lambda_1 = 1 \quad \text{or} \quad \lambda_2 = 3 .$$

Knowing the possible values for a non-zero solution, the modes follow from the algebraic equation when the values of parameter λ are substituted there (one at the time):

$$\lambda_1 = 1: \quad \omega_1 = \frac{3}{L}\sqrt{\frac{S}{\rho A}} \quad \text{and} \quad \begin{bmatrix} 2-1 & -1 \\ -1 & 2-1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{so} \quad (\omega_1, \mathbf{A}_1) = (\frac{3}{L}\sqrt{\frac{S}{\rho A}}, \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}), \leftarrow$$

$$\lambda_2 = 3: \ \ \omega_2 = \frac{3}{L}\sqrt{3\frac{S}{\rho A}} \ \text{ and } \begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \ \text{ so } \ (\omega_2, \mathbf{A}_2) = (\frac{3}{L}\sqrt{3\frac{S}{\rho A}}, \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}).$$ ←