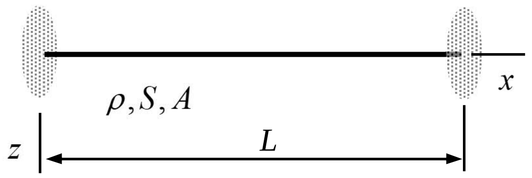
Name____________________________ Student number___________________

Home assignment 3

Find the transverse displacement $w(x,t)$ of the string shown if the initial velocity and displacement at $t=0$ are $\partial w/\partial t=\dot{W}\sin(\pi x/L)$ ($\dot{W}$ is constant) and $w=0$, respectively. Use the continuum model and assume that the string tightening S and the mass per unit length ρA are constants.



Solution

Vibration problem is solved in two steps. First, modal analysis is used to find the generic solution to the equation of motion. The step is based on a trial solution converting the partial differential equation into an ordinary one. Boundary conditions are used find the combinations of the parameters in the solution related with the problem. The outcome are the angular velocity-mode pairs. Second, the series composed of the trial solutions is tuned to match the initial conditions in the mode superposition step.

Let us start with the modal analysis. According to the recipe, the modes are of the form

$$A(x)=\delta\sin(\lambda x)+\gamma\cos(\lambda x)\,,\quad \lambda=\omega\sqrt{\frac{m'}{k'}}$$

where the proper combination of the three parameters δ, γ, λ follow from the boundary conditions. In the present case, both ends are fixed so

$$A(0)=\gamma=0 \quad\text{and}\quad A(L)=\delta\sin(\lambda L)+\gamma\cos(\lambda L)=0\,.$$

A non-zero mode is obtained by selection $\sin(\lambda L)=0$ so $\lambda L = j\pi \quad j\in\{1,2,\ldots\}$. The mode angular velocity pairs (string model $m'=\rho A$ and $k'=S$)

$$A_j(x)=\sin(\lambda_j x) \quad\text{and}\quad \omega_j=\lambda_j\sqrt{\frac{S}{\rho A}} \quad\text{where}\quad \lambda_j L = j\pi \quad j\in\{1,2,\ldots\}\,.$$

Knowing the modes, the particular combination satisfying the initial conditions follows with mode superposition according to

$$u(x,t)=\sum\nolimits_{j\in\{1,2,\ldots\}} A_j(x)[\alpha_j\frac{1}{\omega_j}\sin(\omega_j t)+\beta_j\cos(\omega_j t)]$$

where $\alpha_j = \frac{1}{\mathrm{A}_j^2}\int_0^L A_j(x)hdx$, $\beta_j = \frac{1}{\mathrm{A}_j^2}\int_0^L A_j(x)gdx$ where $\mathrm{A}_j^2 = \int_0^L A_j(x)A_j(x)dx$

Using the initial condition data $g(x) = 0$, $h(x) = \dot{W}\sin(\pi x / L)$ and the modes (orthogonal as can be verified, e.g., graphically, by hand calculation, or with Mathematica)

$$A_1(x) = \sin(\pi\frac{x}{L}),\quad A_1(x) = \sin(2\pi\frac{x}{L}),\ \cdots\ , A_j(x) = \sin(j\pi\frac{x}{L})$$

of which the first is of the same form as the given initial velocity, one obtains that $\alpha_1 \neq 0$ the remaining coefficients being zeros. The value of the coefficient

$$\alpha_1 = \frac{\int_0^L A_1(x)hdx}{\int_0^L A_1(x)A_1(x)dx} = \frac{\int_0^L \sin(\pi\frac{x}{L})\dot{W}\sin(\pi\frac{x}{L})dx}{\int_0^L \sin(\pi\frac{x}{L})\sin(\pi\frac{x}{L})dx} = \dot{W}\ .$$

Using the series with only the non-zero term and the relationship $\lambda_1 = \pi / L = \omega_1\sqrt{\rho A / S}$ gives

$$w(x,t) = A_1(x)\alpha_1\frac{1}{\omega_1}\sin(\omega_1 t) = \dot{W}\sin(\pi\frac{x}{L})\frac{L}{\pi}\sqrt{\frac{\rho A}{S}}\sin(\frac{\pi}{L}\sqrt{\frac{S}{\rho A}}t)\ .$$