Mathematics for Economists: Lecture 7

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This lecture covers

1. Existence of solutions to optimization problems
2. Simplest constrained optimization
3. Optimization subject to an equality constraint: first-order conditions
4. Examples of equality constrained optimization problems

Existence of optimal choices: why care?

Example (1 is the largest natural number)

Proof.

- Denote the largest natural number (i.e. strictly positive integer) by x.
- Since x is a natural number, also x^2 is a natural number.
- Since x is the largest natural number, we have:

$$x \geq x^2.$$

- Dividing both sides by x (a positive number since it is a natural number), we get

$$1 \geq x.$$

- Since all natural numbers are larger than or equal to 1, the claim follows.
- Where is the mistake?

□

Existence helps the characterization

- Suppose that you know that a problem has a solution
- Suppose you know that a single point satisfies first-order necessary conditions
- Do you need to worry about second-order conditions at the critical point?
- Hence it is important to find general enough conditions that guarantee the existence of a solution
- Weierstrass' theorem is pretty good at this

Existence of optimal choices

- You need to check conditions
  1. On the domain (feasible set) of the problem
  2. On the objective function
- Feasible set F bounded
  1. Can you fit the feasible set in a large enough box of finite size?
  2. If $F \in \mathbb{R}^n$, show that $F \subset [-M, M]^n$ for some $M < \infty$.
  3. E.g. $F = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{p} \cdot \boldsymbol{x} \leq w, \boldsymbol{x} \geq 0\}$ for some strictly positive price vector $\boldsymbol{p}$ satisfies $x_i \leq \frac{w}{\min_i p_i}$ for all i.
- Feasible set F closed
  1. If $\boldsymbol{x}_n \to \boldsymbol{x}$ and $\boldsymbol{x}_n \in F$ for all n, then $\boldsymbol{x} \in F$.
  2. Constraint given by continuous functions $F = \{\boldsymbol{x} | g(\boldsymbol{x}) \leq 0\}$ for a continuous $g : \mathbb{R}^n \to \mathbb{R}^k$.
  3. $F = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{p} \cdot \boldsymbol{x} \leq w, \boldsymbol{x} \geq 0\}$ satisfies this.

Existence of optimal choices

- Objective function continuous
  1. Almost all functions you will encounter in economics are continuous
  2. Utility functions, cost functions etc.
  3. Failure of continuity: demand for price setting firms where all buyers buy from the firm with the lowest price.
- A set $F \subset \mathbb{R}^n$ is called *compact* if it is closed and bounded.
- Once we recall that all sets in $\mathbb{R}$ have a greatest lower bound (infimum) and a lowest upper bound (supremum), we are ready for the main existence result, Weierstrass theorem

Main existence theorem

Theorem (Weierstrass' Theorem)

Suppose f is a continuous function on a compact set F, and

$$M = \sup_{x \in F} f(x), \qquad m = \inf_{x \in F} f(x).$$

Then there exist points $\overline{x}, \underline{x} \in F$ *such that* $f(\overline{x}) = M$ *and* $f(\underline{x}) = m$.

Weierstrass' theorem

Remark

To see that F must be closed and bounded and that f has to be continuous, consider the following examples where one property fails in each case:

1. $f(x) = x$ *and* $F = \mathbb{R}$ *(F not bounded).*
2. $f(x) = x$ *and* $F = \{x : 0 < x < 1\}$*. (F not closed).*
3. $f(x) = x$ *for* $0 \leq x < 1$, $f(1) = 0$ *and* $E = \{x : 0 \leq x \leq 1\}$ *(f not continuous).*

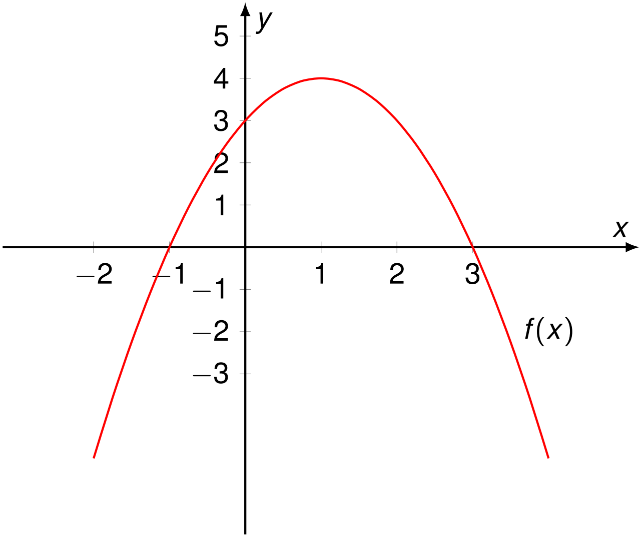
Constrained Optimization: Example

We start with a simplest example of constrained optimization to set up expectations for the more general case to follow.

Example

Consider finding the maximum for $f(x) = 3 + 2x - x^2$ on the feasible set $F = \{x : -\infty < a \leq x \leq b < \infty$.
Since f is continuous and the feasible set F is compact. Therefore Weierstrass' theorem guarantees the existence of a maximizer, i.e. an $x \in F$ such that for all $y \in F$, we have $f(x) \geq f(y)$.
Notice that f is strictly increasing for $x < 1$ and strictly decreasing for $x > 1$.
If $a \leq 1 \leq b$, then the function is maximized at its critical point $x = 1$.
We say that a direction $(x - x_0)$ is feasible from $x_0 \in F$ if for a small Δ, we have $x_0 + \Delta \in F$.

Quadratic function on an interval



Constrained Optimization: Example continued

Linear approximation by the derivative gives:

$$f(x_0 + \Delta) - f(x_0) = f'(x_0)\Delta.$$

If we have a maximum at x_0, then for all feasible direction

$$f'(x_0)\Delta \leq 0.$$

- If $a < x_0 < b$, then we must have $f'(x_0) = 0$ since both directions $\Delta > 0, \Delta < 0$ are feasible.
- If $f'(x_0) > 0$, then $x > x_0$ cannot be feasible if x_0 is a maximum. Therefore $x_0 = b$ if x_0 is the optimal choice and $f'(x_0) > 0$. Similarly, if $f'(x_0) < 0$ and x_0 is the optimum, then $x_0 = a$.

Example: Consumer's problem

- A consumer chooses how to spend her wealth w on two goods x_1, x_2 whose prices are p_1, p_2.
- Continuously differentiable utility, strictly positive marginal utility for both goods.
- The feasible set (budget set) $\{(x_1, x_2)|x_1 \geq 0, x_2 \geq 0, p_1x_1 + p_2x_2 \leq w\}$ is closed (since it is defined by weak inequalities for a continuous constraint function).
- By Weierstrass theorem, a utility maximizing choice exists. What can we say about it?
- Budget constraint must bind: if $p_1\hat{x}_1 + p_2\hat{x}_2 < w$, then $(\hat{x}_1 + h, \hat{x}_2)$ is feasible and:

$$u(\hat{x}_1 + h, \hat{x}_2) > u(\hat{x}_1, \hat{x}_2).$$

- Therefore, we must have $p_1\hat{x}_1 + p_2\hat{x}_2 = w$.
- Can the indifference curve through the optimum intersect the budget line?

Optimization with a single equality constraint

- Local considerations: Let $f : \mathbb{R}^n \to \mathbb{R}$ be the objective function to be maximized
- Suppose the constraints take the form $g(x) = g(x_1, ..., x_n) = 0$.
- In other words, $F = \{x : g(x) = 0\}$. We write the maximization problem often as:

$$\max_x f(x)$$
$$\text{subject to } g(x) = 0.$$

Optimization with a single equality constraint

- A solution to this problem finds point $\hat{\boldsymbol{x}}$ such that $f(\hat{\boldsymbol{x}}) \geq f(\boldsymbol{x})$ for all $\boldsymbol{x} \in F$.
- What can we say about such an $\hat{\boldsymbol{x}}$?
- At this point, we do not know if it exists. If it exists, and f is differentiable, then for small Δ,

$$f(\hat{\boldsymbol{x}} + \Delta(\boldsymbol{x} - \hat{\boldsymbol{x}})) - f(\hat{\boldsymbol{x}}) = Df(\hat{\boldsymbol{x}})(\boldsymbol{x} - \hat{\boldsymbol{x}})\Delta \leq 0$$

  for all feasible directions $(\boldsymbol{x} - \hat{\boldsymbol{x}})$.
- But how do we know which directions are feasible?

Optimization with a single equality constraint

- Assume that the function g defining the constraint is also differentiable.
- To find the feasible directions, we go back to implicit function theorem.
- If $\hat{x} \in F$ and $\frac{\partial g}{\partial x_i}(\hat{\boldsymbol{x}}) \neq 0$ for some $i \in \{1, ..., n\}$, then we can find a write $x_i = h(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) =: h(\boldsymbol{x}_{-i})$ in a neighborhood of $\hat{\boldsymbol{x}}_{-i}$ so that

$$g(h(\boldsymbol{x}_{-i}), \boldsymbol{x}_{-i}) = 0.$$

- Notice that it is not possible to use the implicit function theorem at a critical point of the constraint function.
- Therefore, we must assume that $Dg(\hat{\boldsymbol{x}}) \neq 0$.
- We call this the constraint qualification and we will see different versions of this in more complex situations with multiple constraints.

Optimization with a single equality constraint

- Since the function g is at constant value in the feasible set, we have for all feasible directions $(\boldsymbol{x} - \hat{\boldsymbol{x}})$:

$$\nabla g(\hat{\boldsymbol{x}})(\boldsymbol{x} - \hat{\boldsymbol{x}}) = 0.$$

- Notice also that if $(\boldsymbol{x} - \hat{\boldsymbol{x}})$ is feasible, then also $-(\boldsymbol{x} - \hat{\boldsymbol{x}})$ is feasible. From the linear approximation above, this means that for all feasible directions,

$$Df(\hat{\boldsymbol{x}})(\boldsymbol{x} - \hat{\boldsymbol{x}}) = 0.$$

- But therefore we have shown that at optimum $\hat{\boldsymbol{x}}$,

$$\nabla f(\hat{\boldsymbol{x}}) = \mu \nabla g(\hat{\boldsymbol{x}}).$$

- We have the following necessary condition for a constrained optimum at $\hat{\boldsymbol{x}}$:
    1. the gradient of the objective function must be a scalar multiple of the gradient of the constraint function at the optimum
    2. the choice must be feasible, i.e. $g(\hat{\boldsymbol{x}}) = 0$
    3. we have assumed constraint qualification at optimum

Lagrangean function

- The previous discussion motivates the following function that incorporates the constraints into an augmented objective function called the Lagrangean function.
- For a constrained optimization problem, we define the following function of $n+1$ variables:

$$\mathcal{L}(\boldsymbol{x}, \mu) = f(\boldsymbol{x}) - \mu g(\boldsymbol{x}).$$

- We call the new variable μ the Lagrange multiplier. We will give it a good economic interpretation later in the course.
- We are interested in the critical points of this augmented function. Therefore we look for $(\hat{\boldsymbol{x}}, \hat{\mu})$ such that

$$\frac{\partial \mathcal{L}}{\partial x_i}(\hat{\boldsymbol{x}}, \hat{\mu}) = \frac{\partial f}{\partial x_i}(\hat{\boldsymbol{x}}) - \hat{\mu}\frac{\partial g}{\partial x_i}(\hat{\boldsymbol{x}}) = 0 \text{ for all } i,$$

$$\frac{\partial \mathcal{L}}{\partial \mu}(\hat{\boldsymbol{x}}, \hat{\mu}) = g(\hat{\boldsymbol{x}}) = 0.$$

Figure: Consumer's problem on a budget line

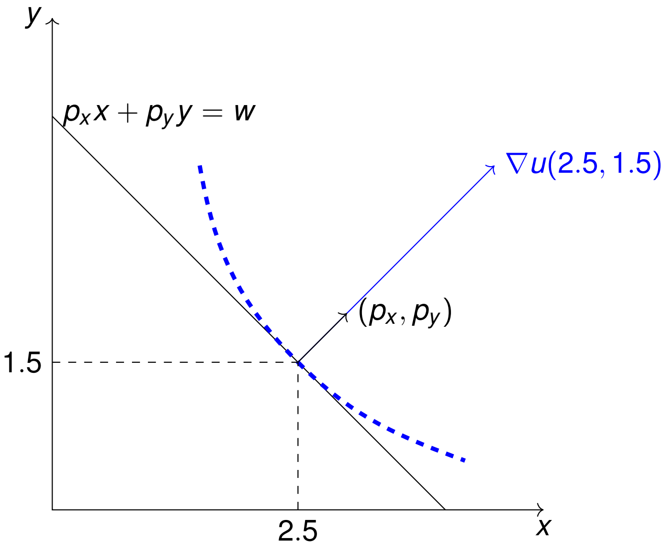
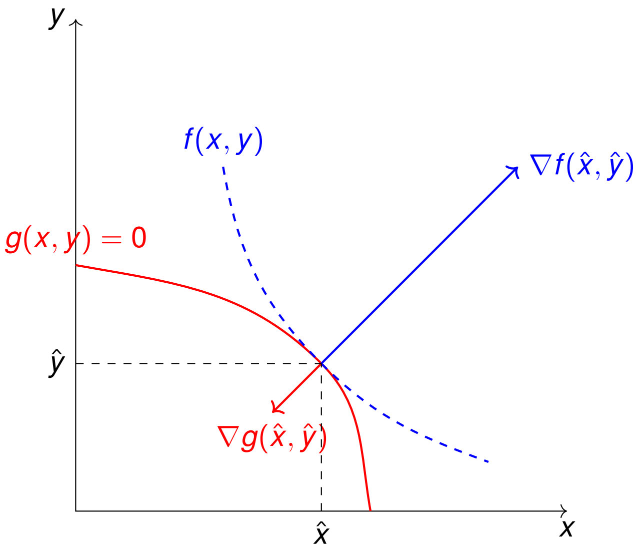


Figure: Single equality constraint



Optimization with a single equality constraint

Find the minima and maxima of $f(x, z) = x + z^2$ subject to constraints

$$x^2 + z^2 = 1$$

Form the Lagrangean:

$$\mathcal{L}(x, z, \mu) = x + z^2 - \mu(x^2 + z^2 - 1)$$

Differentiate to get the first-order conditions (FOC):

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2\mu x = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2z - 2\mu z = 0 \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = 1 - x^2 - z^2 = 0 \tag{3}$$

Optimization with a single equality constraint

- The second FOC gives:

$$z(2 - 2\mu) = 0$$

- therefore either $z = 0$, or $\mu = 1$. Consider first the possibility that $z = 0$. In that case, (3) implies that $x = \pm 1$. We get two critical points from (1):

$$\left(x = 1, z = 0, \mu = \tfrac{1}{2}\right) \text{ and } \left(x = -1, z = 0, \mu = -\tfrac{1}{2}\right)$$

- If $\mu = 1$, (1) implies that $x = \frac{1}{2}$. By substituting into (3) we get the critical points:

$$\left(x = \tfrac{1}{2}, z = \tfrac{\sqrt{3}}{2}, \mu = 1\right) \text{ and } \left(x = \tfrac{1}{2}, z = -\tfrac{\sqrt{3}}{2}, \mu = 1\right)$$

Optimization with a single equality constraint

- As a result, we have four critical points for the Lagrangean. Draw the constraint set and level curves for the objective function to get a guess of the classification of the critical points.
- Can you show the existence of a maximum? Which of the local maxima is the global maximum?

Optimization with multiple equality constraints

Consider next the case, where we have k equality constraints $g(\boldsymbol{x}) = (g_1(\boldsymbol{x}), ..., g_k(\boldsymbol{x})) : \mathbb{R}^n \to \mathbb{R}^k$. In this case, we have the problem:

$$\max_{\boldsymbol{x}} f(\boldsymbol{x})$$

$$\text{subject to } g_1(\boldsymbol{x}) = 0,$$

$$g_2(\boldsymbol{x}) = 0,$$

$$\vdots$$

$$g_k(\boldsymbol{x}) = 0.$$

Form the Lagrangean now with k constraints as a function of $n + k$ variables:

$$\mathcal{L}(\boldsymbol{x}, \mu_1, ..., \mu_k) = f(\boldsymbol{x}) - \sum_{j=1}^{k} \mu_j g_j(\boldsymbol{x}).$$

Optimization with multiple equality constraints

We can proceed exactly as before to use the linear approximations to characterize the feasible directions from $\hat{\boldsymbol{x}}$ as $\{(\boldsymbol{x}-\hat{\boldsymbol{x}}) : Dg(\hat{\boldsymbol{x}})(\boldsymbol{x}-\hat{\boldsymbol{x}})\} = 0$.

Since the objective function cannot increase at the optimum in any feasible direction, we have that

$$Df(\hat{\boldsymbol{x}})(\boldsymbol{x}-\hat{\boldsymbol{x}}) = 0 \text{ whenever } Dg(\hat{\boldsymbol{x}})(\boldsymbol{x}-\hat{\boldsymbol{x}}) = 0.$$

If $Dg(\hat{\boldsymbol{x}})$ has full rank, then this is equivalent to requiring that $Df(\hat{\boldsymbol{x}})$ and $Dg_j(\hat{\boldsymbol{x}})$ must be linearly dependent. Since we assume that $Dg(\hat{\boldsymbol{x}})$ has full rank, this means that there must exist $(\mu_1, ..., \mu_k)$ such that

$$\nabla f(\hat{\boldsymbol{x}}) = \sum_{j=1}^{k} \mu_j \nabla g_j(\hat{\boldsymbol{x}}).$$

Optimization with multiple equality constraints

Hence we can summarize the three necessary conditions for local maximum:

i) Gradient alignment: $\nabla f(\hat{\boldsymbol{x}}) = \sum_{j=1}^{k} \mu_j \nabla g_j(\hat{\boldsymbol{x}})$,

ii) Constraint holds: $g(\hat{\boldsymbol{x}}) = 0$,

iii) Constraint qualification: $Dg_1(\hat{\boldsymbol{x}}), ..., Dg_k(\hat{\boldsymbol{x}})$ are linearly independent.

The first two can be achieved by requiring that $(\hat{\boldsymbol{x}}, \hat{\mu}_1, ..., \hat{\mu}_k)$ be a critical point of the Lagrangean.

Optimization with multiple equality constraints: an example

- Consider the problem of maximizing

$$f(x, y, z) = xz + yz$$

subject to:

$$g_1(x, y, z) = y^2 + z^2 - 1$$
$$g_2(x, y, z) = xz - 3$$

1. Find the critical points of f subject to constraints $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$.
2. How would you determine which of the critical points are local minima and which are local maxima?

Optimization with multiple equality constraints: an example

1. Find first the critical points of the Lagrangean

$$\mathcal{L}(x, y, z, \mu_1, \mu_2) = xz + yz - \mu_1(y^2 + z^2 - 1) - \mu_2(xz - 3)$$

2. First-order conditions:

$$\frac{\partial \mathcal{L}}{\partial x} = z - \mu_2 z = 0 \tag{4}$$
$$\frac{\partial \mathcal{L}}{\partial y} = z - 2\mu_1 y = 0 \tag{5}$$
$$\frac{\partial \mathcal{L}}{\partial z} = x + y - 2\mu_1 z - \mu_2 x = 0 \tag{6}$$
$$\frac{\partial \mathcal{L}}{\partial \mu_1} = y^2 + z^2 - 1 = 0 \tag{7}$$
$$\frac{\partial \mathcal{L}}{\partial \mu_2} = xz - 3 = 0 \tag{8}$$

Optimization with multiple equality constraints: an example

We need to solve this system of equations to find the critical points. Start with (4), giving

$$z(1-\mu_2)=0, \Leftrightarrow \; z=0 \text{ or } \mu_2=1$$

If $z=0$, then (8) is not true for any x and as a result, we must have$z\neq 0$. Therefore, we can only have $\mu_2=1$ as a candidate solution. The second FOC (5) gives

$$y-2\mu_1 z=0, \; \Leftrightarrow y=\frac{z}{2\mu_1}.$$

Optimization with multiple equality constraints: an example

Plug in the solutions for y and μ_2 into (6) :

$$\frac{z}{2\mu_1} - 2\mu_1 z = 0 \;\Leftrightarrow\; z\left(\frac{1}{2\mu_1} - 2\mu_1\right) = 0.$$

We already know that $z \neq 0$, and therefore

$$\frac{1}{2\mu_1} - 2\mu_1 = 0 \;\Leftrightarrow\; 4\mu_1^2 = 1 \;\Leftrightarrow\; \mu_1 = \pm\frac{1}{2}$$

Optimization with multiple equality constraints: an example

We have now solved for possible Lagrange multipliers μ_1 ja μ_2, i.e. we have:

$$\mu_1 = \pm\frac{1}{2} \text{ and } \mu_2 = 1$$

Optimization with multiple equality constraints: an example

To get the values of the choice variables, plug in the values of the multipliers into (6) to get:

$$y = \pm z.$$

Substituting into (7), we get (by squaring):

$$2z^2 - 1 = 0 \Leftrightarrow z = \pm\frac{1}{\sqrt{2}}$$

The fifth FOC (8) gives:

$$x = \frac{3}{z},$$

or $x = 3\sqrt{2}$ if $z = \frac{1}{\sqrt{2}}$ and $x = -3\sqrt{2}$ if $z = -\frac{1}{\sqrt{2}}$. We have now found all that we need for the critical points of f subject to the constraints.

Optimization with multiple equality constraints: an example

If $z = \frac{1}{\sqrt{2}}$, then $x = 3\sqrt{2}$, $y = \pm z$. This yields two critical points (x, y, z):

$$1 : \left(3\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$2 : \left(3\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

If $z = -\frac{1}{\sqrt{2}}$, then $x = -3\sqrt{2}$, $y = \pm z$. This gives also two critical points (x, y, z):

$$3 : \left(-3\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$4 : \left(-3\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

Optimization with multiple equality constraints: an example

We know that for all critical points, $\mu_2 = 1$, and we can check the sign of μ_1 from FOC (5). After this, we have all the critical points of the problem as:

Critical points for the problem (x, y, z, μ_1, μ_2):

$$1: \left(3\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}, 1\right)$$

$$2: \left(3\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{2}, 1\right)$$

$$3: \left(-3\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{2}, 1\right)$$

$$4: \left(-3\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}, 1\right)$$

Next Lecture

- Optimization with inequality constraints
- Economic examples of constrained optimization