Mathematics for Economists
ECON-C1000
Spring 2021
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Problem Set 5: Due May 20, 2021

1. Solve the following maximization problem:

$$\max_{x,y} xy$$

subject to

$$\begin{aligned} x + y &\leq 100 \\ x &\leq 40 \\ x, y &\geq 0 \end{aligned}$$

2. A consumer lives for two periods $t \in \{1, 2\}$ and gets utility $u_t(c_t)$ from consuming $c_t \geq 0$ in period t so that her total utility is $u(c_1, c_2) = u_1(c_1) + u_2(c_2)$. In the first period, she has income $w_1 \geq 0$ and in the second, she has income $w_2 \geq 0$. Assume that $u_t'(c_t) > 0$ and $u_t''(c_t) < 0$ for $t \in \{1, 2\}$.

   (a) What is the feasible set of consumptions (c_1, c_2) for a consumer that can save income (at zero interest) but cannot borrow against future income?

   (b) What is the budget set if the consumer can also borrow (at zero interest) against future income so that any borrowings in $t = 1$ must be paid back from $t = 2$ income?

   (c) For the case where the two utility functions are identical so that $u_1(c_t) = u_2(c_t) = u(c_t)$, set up the constrained optimization problem and solve it when borrowing and saving is possible.

   (d) How does your answer change if the consumer is impatient so that $u_2(c) = \delta u_1(c)$ for some $0 < \delta < 1$?

   (e) When can you have different marginal utilities of consumption across the two periods at optimum in the case where the utility functions are identical and you can only save?

3. Continue with he same setting and now assume that $u(c_1, c_2) = \ln c_1 + \delta \ln c_2$ with $0 < \delta < 1$ for $c_1, c_2 > 0$ and $u(c_1, c_2) = -\infty$ if $c_t = 0$ for some t. Assume that saving and borrowing is possible at interest rate r so that by saving s in $t = 1$, you get back $(1+r)s$ in $t = 2$. If you borrow b in $t = 1$, you must pay back $(1+r)b$ in $t = 2$. (Note that $w_1 - c_1$ is your savings (if it is positive) and borrowing (if it is negative)).

(a) Set up the consumer's problem and its Lagrangean. In particular, derive a single budget constraint for the consumptions. Derive the first-order K-T conditions.

(b) Argue that the K-T conditions are sufficient so that any solution to the K-T conditions solves the maximization problem.

(c) Determine which constraints must bind and solve the problem for optimal consumptions (c_1^*, c_2^*).

(d) When is $c_1^* < c_2^*$? Does this depend on the difference $w_1 - w_2$?

4. Continue with the utility function from the previous problem, but assume now (realistically) that you can save at interest rate $\underline{r}$ and borrow at $\overline{r}$ with $\overline{r} > \underline{r}$.

   (a) Draw the budget. set for the consumer in the (c_1, c_2) coordinates.

   (b) Find bounds for the consumer's MRS_{c_1,c_2} at optimum based on the picture (recall that utility is unboundedly negative as $c_t \to 0$).

   (c) What is the optimality condition for the consumers that save? What is the optimality condition for those who borrow?

   (d) What can be said about the MRS of those consumers that neither save nor borrow?

   (e) Solve the optimization problem.

5. Which of the following claims are true and which are false. For false statements, give a counterexample, for true ones, provide a proof.

   (a) The only functions that are both concave and convex on $\mathbb{R}^n$ are affine functions of the form $f(x) = c + b \cdot x$ for some vector $b \in \mathbb{R}^n$ and. a constant $c \in \mathbb{R}$.

   (b) The only functions in $\mathbb{R}^n$ for $n > 1$ that are both quasiconcave and quasiconvex are affine functions.

   (c) Minimizing the distance from a point to a plane in $\mathbb{R}^3$ can be written as a problem of maximizing a concave function on a convex set.

   (d) If $f(x)$ is a concave function on $\mathbb{R}^n$, then for all increasing and concave $g(y)$, the composite function $g(f(x))$ is also concave. This is something that can be done directly from the definition of concavity, but you may assume that f is twice differentiable if you want.

6. A sleepy father likes to eat, play with his children and sleep. Let $f \geq 0$ denote the amount of food that the father eats, $c \geq 0$ the amount of time with children and $s \geq 0$ the amount of sleep. His utility function is increasing in all three components and for simplicity, assume that it takes the form (with strictly positive α_i):

$$u(f, c, s) = 2\sqrt{f} + 2\sqrt{c} + 2\sqrt{s}.$$

   (a) Show that this is a strictly concave function on non-negative vectors in $(f, c, s) \geq (0, 0, 0)$. Therefore any point satisfying the first-order conditions on any convex feasible set is a global maximum.

(b) Unfortunately to get food, the father must work and working is away from either sleep or playing time with children. What is the budget constraint for the father if he has 24 hours of total time and the wage (in terms of units of food per hour) is $w > 0$?

(c) What is the feasible set? Is it defined by some quasiconvex constraint functions?

(d) Write the maximization problem, the Lagrangean and the first-order conditions for the problem.

(e) Argue that the non-negativity constraints are not binding but the budget constraint is binding.

(f) Solve for the unique point satisfying the necessary conditions. Are the conditions for Weierstrass' theorem satisfied? Do you need to check the definiteness of the bordered Hessian to conclude that you have found a maximum?