Mathematics for Economists

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Problem set 4 Solutions:

Question 1:

$$\max_{x,y} xy$$

$$subject\ to: \quad x + y \leq 100$$

$$x \leq 40$$

$$x, y \geq 0$$

We first form the Lagrangian:

$$L = xy - \lambda_1(x + y - 100) - \lambda_2(x - 40) + \lambda_3 x + \lambda_4 y = 0$$

The first order conditions:

$$\frac{\partial L}{\partial x} = 0 \Rightarrow y - \lambda_1 - \lambda_2 + \lambda_3 = 0 \quad (1)$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow x - \lambda_1 + \lambda_4 = 0 \quad (2)$$

$$\lambda_1(x + y - 100) = 0 \quad (3)$$

$$\lambda_2(x - 40) = 0 \quad (4)$$

$$\lambda_3 x = 0\,, \lambda_4 y = 0$$

$$x + y \leq 100, \quad x \leq 40, \quad x, y \geq 0$$

If $x, y > 0$, then $\lambda_3, \lambda_4 = 0$. Putting this into (1) and (2) we have:

$$x = \lambda_1$$

$$y - x = \lambda_2$$

Using (3) we have:

$$x(x + y - 100) = 0$$

And $x > 0$, so first condition is binding and:

$$x + y = 100 \quad (5)$$

Using (4)

$$(y-x)(x-40)=0 \Rightarrow \begin{cases} y=x \quad not\ valid\ because\ of\ (5)\ and\ the\ fact\ that\ x \le 40 \\ x=40\ valid, so\ the\ second\ condition\ is\ binding \end{cases}$$

So $x^* = 40$. And using (5) we have

$$y^* = 100 - 40 = 60$$

Question 2:

a) The consumer can save in the first period but cannot borrow so the constraints are:

$$0 \le c_1 \le w_1 \quad and \quad 0 \le c_2 + c_1 \le w_2 + w_1$$

b) Now the consumer can borrow at the first stage:

$$0 \le c_1 \le w_1 + b \quad and \quad 0 \le c_2 + c_1 \le w_2 + w_1$$

c) The problem in the case of the identical utility functions:

$$\max_{c_1, c_2} u(c_1) + u(c_2)$$
$$\text{s.t: } c_1 \le w_1 + b$$
$$c_2 + c_1 \le w_2 + w_1$$

First we form the Lagrangian:

$$L = u(c_1) + u(c_2) - \lambda_1(c_1 - w_1 - b) - \lambda_2(c_1 + c_2 - w_1 - w_2)$$

Now we write the focs:

$$\frac{\partial L}{\partial c_1} = u'(c_1) - \lambda_1 - \lambda_2 = 0$$
$$\frac{\partial L}{\partial c_2} = u'(c_2) - \lambda_2 = 0$$
$$\lambda_1(c_1 - w_1 - b) = 0$$
$$\lambda_2(c_1 + c_2 - w_1 - w_2) = 0$$
$$c_1 \le w_1 + b$$
$$c_2 \le w_2 + w_1 - c_1$$
$$c_1, c_2 > 0$$
$$\lambda_1, \lambda_2 \ge 0$$

According to assumptions, $u'(c_t) > 0$ so $\lambda_2 > 0$ so the second constraint is binding and:

$$c_1 + c_2 = w_1 + w_2$$

If we assume that $\lambda_1 \ne 0$ so the first constraint is also binding and $c_1 = w_1 + b$

And

$$c_2 = w_2 - b$$

In the case of $w_2 < b$, c_2 will be negative and it is not a valid solution so $\lambda_1 = 0$, then from focs:

$$u'(c_2) = u'(c_1)$$

And because u' is strictly increasing $c_1 = c_2 = c$

So

$$c^* = \frac{w_1 + w_2}{2}$$

d)

Now the problem is:

$$\max_{c_1, c_2} u(c_1) + \delta u(c_2)$$
$$\text{s.t: } c_1 \leq w_1 + b$$
$$c_2 \leq w_2 + w_1 - c_1$$

All of the steps are the same as the part a and at the end:

$$u'(c_1) = \delta u'(c_2)$$

Considering the facts that:

$$\delta \in (0,1) \;\; and \;\; u''(c) < 0$$

We conclude that:

$$c_1 > c_2$$

e)

According to part d when the consumer is impatient and $u_2(c) = \delta u_1(c)$ and $\delta \in (0,1)$, marginal utility of consumer in the second period will be lower than the one in the first period.

Question 3:

$$\max_{c_1, c_2} \ln(c_1) + \delta \ln(c_2)$$
$$c_2 \leq w_2 + (1+r)(w_1 - c_1)$$

a)

$$L = \ln(c_1) + \delta \ln(c_2) - \mu(c_2 - w_2 - (1+r)(w_1 - c_1))$$
$$\frac{\partial L}{\partial c_1} = \frac{1}{c_1} - \mu(1+r) = 0$$
$$\frac{\partial L}{\partial c_2} = \frac{\delta}{c_2} - \mu = 0$$
$$\mu\big(c_2 - w_2 - (1+r)(w_1 - c_1)\big) = 0$$
$$c_2 \leq w_2 + (1+r)(w_1 - c_1)$$
$$c_1, c_2 > 0$$

b) The constraint function is linear so the feasible set is convex. Moreover, the utility function is strictly increasing and quasiconcave so the first order K-T conditions are sufficient to solve the problem.

c)

$$\frac{\partial L}{\partial c_2} = \frac{\delta}{c_2} - \mu = 0 \Rightarrow \mu \neq 0 \Rightarrow budget\ constraint\ is\ binding$$

Moreover

$$\frac{\delta}{c_2} = \mu = \frac{1}{c_1(1+r)} \Rightarrow c_2 = c_1\delta(1+r)$$

Putting this into the budget constraint we have:

$$c_1\delta(1+r) = w_2 + (1+r)(w_1 - c_1) \Rightarrow c_1^* = \frac{w_2 + w_1(1+r)}{(1+\delta)(1+r)}$$

$$c_2^* = \frac{\delta(w_2 + w_1(1+r))}{(1+\delta)}$$

d)

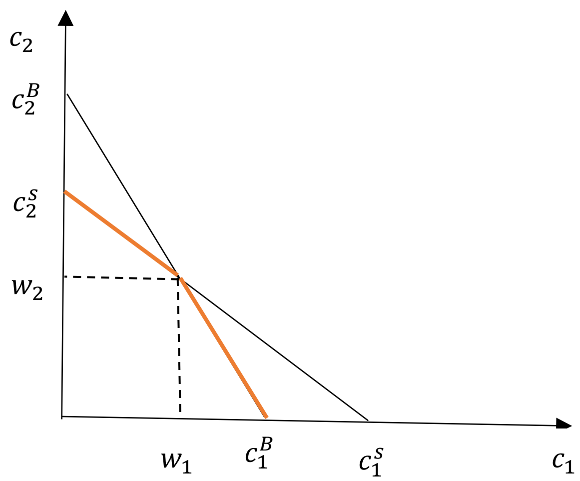
$$c_2 = c_1\delta(1+r) \; and \;\; c_1, c_2 > 0$$

If $c_1^* < c_2^*$, then $\delta(1+r) > 1$

Question 4:

a)

$$\max_{c_1,c_2} \ln(c_1) + \delta \ln(c_2)$$
$$w_1 - c_1 < 0 \rightarrow c_2 \le w_2 + (1+\bar{r})(w_1 - c_1)$$
$$w_1 - c_1 > 0 \rightarrow c_2 \le w_2 + (1+\underline{r})(w_1 - c_1)$$



b)

$$1 + \underline{r} \le MRS_{(c_1=w_1, c_2=w_2)} \le 1 + \bar{r}$$

Otherwise it is not going to be optimum.

C,d)

$$if \;\; 1 + \underline{r} < MRS(c_1 = w_1, c_2 = w_2) < 1 + \bar{r}$$

According to this condition you are at the optimum and there is no way (by saving or borrowing) you could increase your utility.

$$if \quad MRS(c_1 = w_1, c_2 = w_2) > 1 + \bar{r}$$

High MRS means that we prefer to borrow money to increase c_1 and decrease c_2. According to the figure in part a, when MRS is high at the point $(c_1 = w_1, c_2 = w_2)$ we are at the bottom of the $1 + \bar{r}$ budget constraint line. We can increase our utility by borrowing and decreasing c_2 to increase the consumption in period one c_1.

$$if \quad 1 + \underline{r} > MRS(c_1 = w_1, c_2 = w_2)$$

Low MRS means that we want to save money to increase c_2 and decrease c_1. (we are at the bottom of the $1 + \underline{r}$ budget line)

e)

Before solving this problem we should consider that this is exactly the same as question 3. The only difference is that now we should solve the problem for two different cases with two different budget lines, so at the end one of them will be our final solution for the optimization problem. According to the results of the question 3 we have:

For points where $c_1 < w_1 \ and \ c_2 > w_2$:

$$c_{1,1}^* = \frac{w_2 + w_1(1 + \bar{r})}{(1 + \delta)(1 + \bar{r})}$$

$$c_{2,1}^* = \frac{\delta(w_2 + w_1(1 + \bar{r}))}{(1 + \delta)}$$

For points where $c_1 > w_1 \ and \ c_2 < w_2$:

$$c_{1,2}^* = \frac{w_2 + w_1(1 + \underline{r})}{(1 + \delta)(1 + \underline{r})}$$

$$c_{2,2}^* = \frac{\delta(w_2 + w_1(1 + \underline{r}))}{(1 + \delta)}$$

Only one of these solutions will be valid (according to the conditions over c_1, c_2) at the end.

Question 5:

a) For convex functions we have:

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

And for concave functions:

$$f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta)f(y)$$

So if the function f is convex and concave at the same time then:

$$f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta)f(y)$$

To prove that the only function that has the above characteristics is the affine function we should prove that:

1- $f(\alpha x) = \alpha f(x)$ and
2- $f(x + y) = f(x) + f(y)$

Without loss of generality we assume that $f(0) = 0$ (if not we can easily assume function g, where $g(x) = f(x) - f(0)$ and we need the lemma that if function f is both concave and convex then the function $f(x) - f(0)$ is both convex and concave and then prove the characteristics for function g.)

1- $f(\alpha x) = f(\alpha x + (1-\alpha) * 0) = \alpha f(x) + (1-\alpha)f(0) = \alpha f(x)$
2- $f(x + y) = f\left(\frac{1}{2}.2x + \frac{1}{2}.2y\right) = \frac{1}{2}f(2x) + \frac{1}{2}f(2y) = f(x) + f(y)$

So f is an affine function.

b) For simplicity we start with n=1. According to the definitions of the quasiconcave and quasiconvex functions:

Quasi concave functions: $f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}$
Quasi convex functions: $f(\alpha x + (1-\alpha)y) \leq \max\{f(x), f(y)\}$

So any increasing or decreasing function f in R_+ is both quasiconcave and quasiconves, such as:

$$f(x) = x^2 \quad or \quad f(x) = e^x \quad ...$$

Now for the case of n=2, consider any strictly monotone function of an affine function such as $(y - x)^3$. This is a composite function $f(g(x))$ where g is an affine function (both quasi concave and quasi convex) and f is strictly increasing, so we can easily conclude that the function $f(g(x))$ is also both quasi convex and quasi concave.

c) Assume the point to be:

$$p_0 = (x_0, y_0, z_0)$$

And the plane N to be:

$$N: Ax + By + Cz + D = 0$$

So the optimization problem is:

$$\min_{(x,y,z)\in N} (x-x_0)^2+(y-y_0)^2+(z-z_0)^2$$
$$subject\ to \qquad Ax+By+Cz+D=0$$

We can equivalently write it as:

$$\max_{(x,y,z)\in N} -[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]$$
$$subject\ to \qquad Ax+By+Cz+D=0$$

Now the objective function is a concave function and the constraint is the set of all the points on a plain which is obviously a convex set.

d)

$f(x)$ is a concave so

$$f(\lambda_1 x_1+\cdots+\lambda_n x_n) > \lambda_1 f(x_1)+\cdots+\lambda_n f(x_n)$$

$$g(y)\ is\ increasing\ and\ concave\ so\ \ g'(x)\geq 0\ and\ \ g''(x)\leq 0$$

now we start with the composite function:

$$g\big(f(x)\big)=g\big(f(\lambda_1 x_1+\cdots+\lambda_n x_n)\big)$$

since g is an increasing function and f is concave:

$$g\big(f(\lambda_1 x_1+\cdots+\lambda_n x_n)\big) > g\big(\lambda_1 f(x_1)+\cdots+\lambda_n f(x_n)\big)$$

and because g is concave

$$g\big(\lambda_1 f(x_1)+\cdots+\lambda_n f(x_n)\big) > \lambda_1 g\big(f(x_1)\big)+\cdots+\lambda_n g(f(x_n))$$

so $g\big(f(x)\big)$ is a concave function.

Question 6:

a)

We form the Hessian matrix of the function u:

$$u(f,c,s)=2\sqrt{f}+2\sqrt{c}+2\sqrt{s}$$

So

$$H_u = \begin{bmatrix} \frac{\partial^2 u}{\partial f^2} & \frac{\partial^2 u}{\partial f \partial c} & \frac{\partial^2 u}{\partial f \partial s} \\ \frac{\partial^2 u}{\partial c \partial f} & \frac{\partial^2 u}{\partial c^2} & \frac{\partial^2 u}{\partial c \partial s} \\ \frac{\partial^2 u}{\partial f \partial s} & \frac{\partial^2 u}{\partial s \partial c} & \frac{\partial^2 u}{\partial s^2} \end{bmatrix} = \begin{bmatrix} -\frac{\frac{1}{2}}{f^{\frac{3}{2}}} & 0 & 0 \\ 0 & -\frac{\frac{1}{2}}{c^{\frac{3}{2}}} & 0 \\ 0 & 0 & -\frac{\frac{1}{2}}{s^{\frac{3}{2}}} \end{bmatrix}$$

Which is obviously negative definite for all the (f, c, s) in the domain, so the function u is strictly concave.

b)

So the time of the father is allocated to three different tasks:

$$s + c + h_w = 24$$

Where h_w is the working time. Moreover the income should be equal to the expenses so:

$$h_w . w \geq f$$

So over all

$$(24 - s - c). w - f \geq 0$$

c)

The feasible set is:

$$g(f, c, s) = (24 - s - c). w - f \geq 0$$

$$0 \leq s \leq 24$$

$$0 \leq c \leq 24$$

$$0 \leq f \leq 24w$$

d)

$$\max_{f,c,s} 2\sqrt{f} + 2\sqrt{c} + 2\sqrt{s}$$

$$st. (24 - s - c). w - f \geq 0$$

$$s, c, f \geq 0$$

We form the Lagrangian:

$$L = 2\sqrt{f} + 2\sqrt{c} + 2\sqrt{s} + \lambda((24 - s - c). w - f)$$

The first order conditions:

$$\frac{\partial L}{\partial f} = \frac{1}{\sqrt{f}} - \lambda = 0$$

$$\frac{\partial L}{\partial c} = \frac{1}{\sqrt{c}} - \lambda w = 0$$

$$\frac{\partial L}{\partial s} = \frac{1}{\sqrt{s}} - \lambda w = 0$$

$$\lambda[(24 - s - c).w - f] = 0$$

e,f)

from the first three conditions

$$\lambda = \frac{1}{\sqrt{f}} = \frac{1}{w\sqrt{c}} = \frac{1}{w\sqrt{s}} \rightarrow c = s = \frac{f}{w^2}$$

since $\lambda \neq 0$, the budget constraint should bind so

$$[(24 - s - c).w - f = 0$$

using these two constraints, we have:

$$\left(24 - \frac{2f}{w^2}\right)w - f = 0 \rightarrow 24w = f\left(\frac{2}{w} + 1\right)$$

$$f^* = \frac{24w^2}{w+2} \;, c^* = s^* = \frac{24}{w+2}$$

The Weierstrass' theorem is satisfied because the objective function is a continuous function that is defined on a closed interval. Note that since the partial derivatives of the function with respect to f,s and c are infinite at zero, they are not defined there.