COE-C3005 Finite Element and Finite difference methods

1. Consider a piecewise linear approximation $g(x)$ to $f(x)=x^2$ $x\in[0,L]$ on the regular grid of 3 points. Use the weighted residual method

$$\int_0^L \mathbf{N}(\mathbf{N}^{\mathrm{T}}\mathbf{g}-f)dx=0,$$

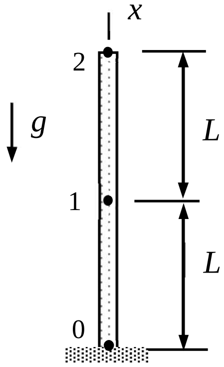
where $\mathbf{N}^{\mathrm{T}}=\{N_0 \quad N_1 \quad N_2\}$ to find the values $\mathbf{g}^{\mathrm{T}}=\{g_0 \quad g_1 \quad g_2\}$ of the approximation at the grid points.

Answer $\mathbf{g}^{\mathrm{T}}=\dfrac{L^2}{24}\{-1 \quad 5 \quad 23\}$

2. Derive the iteration according to the Discontinuous Galerkin Method using (1) a polynomial approximation $a(t)=\alpha_0+\alpha_1 t$ to displacement in a typical time interval of length Δt, and (2) weighted residual expression

$$\int_{t_{i-1}}^{t_i} p(m\ddot{a}+ka)dt+[pm(\dot{a}-h)]_{i-1}-[\dot{p}m(a-g)]_{i-1}=0 \quad \text{where } p\in\{1,t\}.$$

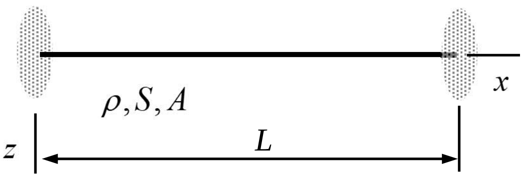
Answer $\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i=\dfrac{2}{12+\alpha^4}\begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1}$ where $\alpha=\sqrt{\dfrac{k}{m}}\Delta t$



3. The bar shown is loaded by its own weight. Determine the displacements at the grid points 1 and 2 using the Finite Element Method. Cross-sectional area A, Young's modulus E, and density ρ of the material are constants.

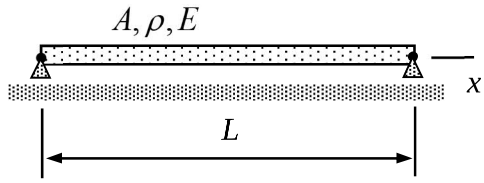
Answer $u_0=0$, $u_1=-\dfrac{3}{2}\dfrac{\rho g L^2}{E}$, $u_2=-2\dfrac{\rho g L^2}{E}$

4. Consider the string of tightening S and mass per unit length ρA shown. Use the Finite Element Method on a regular grid $i\in\{0,1,\ldots,n\}$ to find the angular velocities ω_k of the free vibrations using the solution trial $w_i=a_k(t)\sin(k\pi i/n)$ $k\in\{1,2\ldots,n-1\}$.



Answer $\omega_k=\dfrac{12}{L}\sqrt{\dfrac{S}{A\rho}}n\sin(\dfrac{k\pi}{2n})/\sqrt{2+\cos(\dfrac{k\pi}{n})}$ $k\in\{1,2\ldots,n-1\}$ (exact $\omega_k=\dfrac{k\pi}{L}\sqrt{\dfrac{S}{A\rho}}$)

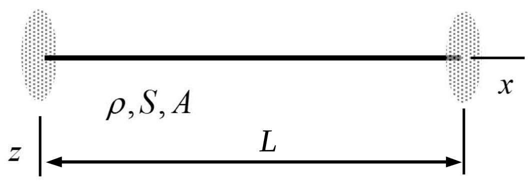
5. A bar is free to move in the horizontal direction as shown. Write the equation system $\mathbf{Ka}+\mathbf{M\ddot{a}}=0$ given by the Finite Element Method on a regular grid with $i\in\{0,1,2\}$. Also, determine the angular velocities and modes of the free vibrations. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Answer $2\frac{EA}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_0\\ u_1\\ u_2\end{Bmatrix}+\rho A\frac{L}{2}\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_0\\ \ddot{u}_1\\ \ddot{u}_2\end{Bmatrix}=0$ and $(\omega,\mathbf{A})_1=(0,\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix})$,

$(\omega,\mathbf{A})_2=(\frac{2}{L}\sqrt{3\frac{E}{\rho}},\begin{Bmatrix}-1\\ 0\\ 1\end{Bmatrix})$, $(\omega,\mathbf{A})_3=(\frac{4}{L}\sqrt{3\frac{E}{\rho}},\begin{Bmatrix}1\\ -1\\ 1\end{Bmatrix})$

6. Consider the string of tightening S and mass per unit length ρA shown. First, use the Finite Element Method with the second order accurate central differences on a regular grid $i\in\{0,1,2\}$ to find the equation of motion of the form $ka+m\ddot{a}=0$. Second, write the iteration equation for a typical time-step of size Δt according to Discontinuous Galerkin Method giving the values of displacement and velocity on the temporal grid.



Answer $4\frac{EA}{L}u_1+\rho AL\frac{1}{3}\ddot{u}_1=0$, $\begin{Bmatrix}a\\ \dot{a}\Delta t\end{Bmatrix}_i=\frac{2}{12+\alpha^4}\begin{bmatrix}6-3\alpha^2 & 6-\alpha^2\\ -6\alpha^2 & 6-3\alpha^2\end{bmatrix}\begin{Bmatrix}a\\ \dot{a}\Delta t\end{Bmatrix}_{i-1}$, $\alpha=\sqrt{12\frac{E}{\rho}}\frac{\Delta t}{L}$

Consider a piecewise linear approximation $g(x)$ to $f(x)=x^2$ $x\in[0,L]$ on the regular grid of 3 points. Use the weighted residual method

$$\int_0^L \mathbf{N}(\mathbf{N}^{\mathrm{T}}\mathbf{g}-f)dx=0,$$

where $\mathbf{N}^{\mathrm{T}}=\{N_0 \quad N_1 \quad N_2\}$ to find the values $\mathbf{g}^{\mathrm{T}}=\{g_0 \quad g_1 \quad g_2\}$ of the approximation at the grid points.

Solution

In the weighted residual method with piecewise linear approximation, the aim is to find the approximation $g=\mathbf{N}^{\mathrm{T}}\mathbf{g}$ so that

$$\int_0^L \mathbf{N}(g-f)dx=0.$$

As the approximation is piecewise continuous, the integral needs to be calculated as the sum of integrals over the line segments between the grid points. The equations given by the three shape functions take the form

$$\int_0^{L/2} N_0(N_0g_0+N_1g_1-f)dx=0,$$

$$\int_0^L N_1(g-f)dx=\int_0^{L/2} N_1(N_0g_0+N_1g_1-f)dx+\int_{L/2}^L N_1(N_1g_1+N_2g_2-f)dx=0,$$

$$\int_0^L N_2(N_1g_1+N_2g_2-f)dx=0.$$

Using $f(x)=x^2$ and the shape function expression

$$N_0=1-2\frac{x}{L} \quad x<\frac{L}{2} \quad \text{and} \quad N_0=0 \quad x>\frac{L}{2},$$

$$N_1=2\frac{x}{L} \quad x<\frac{L}{2} \quad \text{and} \quad N_1=2(1-\frac{x}{L}) \quad x>\frac{L}{2},$$

$$N_2=2\frac{x}{L}-1 \quad x<\frac{L}{2} \quad \text{and} \quad N_1=2(1-\frac{x}{L}) \quad x>\frac{L}{2},$$

The three equations become

$$\int_0^L N_0(N_0g_0+N_1g_1-f)dx=g_0\frac{L}{6}+g_1\frac{L}{12}-\frac{L^3}{96}=0,$$

$$\int_0^L N_1(g-f)dx=\frac{L}{12}g_0+\frac{L}{3}g_1+\frac{L}{12}g_2-\frac{7L^3}{48}=0,$$

$$\int_0^L N_2(N_1 g_1 + N_2 g_2 - f)dx = \frac{L}{12} g_1 + \frac{L}{6} g_2 - \frac{17L^3}{96} = 0 \,.$$

Or when written in matrix form and simplified somewhat

$$\begin{bmatrix} 16 & 8 & 0 \\ 8 & 32 & 8 \\ 0 & 8 & 16 \end{bmatrix} \begin{Bmatrix} g_0 \\ g_1 \\ g_2 \end{Bmatrix} - L^2 \begin{Bmatrix} 1 \\ 14 \\ 17 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} g_0 \\ g_1 \\ g_2 \end{Bmatrix} = \frac{L^2}{24} \begin{Bmatrix} -1 \\ 5 \\ 23 \end{Bmatrix} .$$

Derive the iteration according to the Discontinuous Galerkin Method using (1) a polynomial approximation $a(t)=\alpha_0+\alpha_1 t$ to displacement in a typical time interval of length Δt, and (2) weighted residual expression

$$\int_{t_{i-1}}^{t_i} p(m\ddot{a}+ka)dt+[pm(\dot{a}-h)]_{i-1}-[\dot{p}m(a-g)]_{i-1}=0 \quad \text{where } p\in\{1,t\}.$$

Solution

The simplest time discontinuous Galerkin method for an initial value problems of a second order ordinary differential equations uses the polynomial approximations $a(t)=\alpha_0+\alpha_1 t$ and weighted residual expression of the equations

$$\int_{t_{i-1}}^{t_i} p(m\ddot{a}+ka)dt+[pm(\dot{a}-h)]_{i-1}-[\dot{p}m(a-g)]_{i-1}=0,$$

where the weighting $p\in\{1,t\}$. Considering the time interval $[0,\Delta t]$ so $t_{i-1}=0$ and $t_i=\Delta t$ (just to simplify the manipulations) substituting the approximation $a(t)=\alpha_0+\alpha_1 t$, and writing the weighted residual expression with the two selections of the weighting function gives

$$k(\alpha_0\Delta t+\frac{1}{2}\alpha_1\Delta t^2)+[m(\alpha_1-h)]=0 \quad \text{and} \quad k(\alpha_0\frac{1}{2}\Delta t^2+\alpha_1\frac{1}{6}\Delta t^3)-m(\alpha_0-g)=0.$$

Or, when written in the matrix form

$$\begin{bmatrix} k\frac{1}{2}\Delta t^2-m & k\frac{1}{6}\Delta t^3 \\ k\Delta t & \frac{1}{2}\Delta t^2+m \end{bmatrix}\begin{Bmatrix}\alpha_0\\ \alpha_1\end{Bmatrix}=m\begin{Bmatrix}-g\\ h\end{Bmatrix}.$$

Assuming that the initial conditions are given by the solution to the previous time interval $h=\dot{a}_{i-1}$ and $g=a_{i-1}$ and writing the approximation $a(t)=\alpha_0+\alpha_1 t$ at the end point of the time interval $a_i=\alpha_0+\alpha_1\Delta t$ and $\dot{a}_i=\alpha_1$:

$$\begin{Bmatrix}a\\ \dot{a}\end{Bmatrix}_i=\begin{bmatrix}1 & \Delta t\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\alpha_0\\ \alpha_1\end{Bmatrix} \quad\Leftrightarrow\quad \begin{Bmatrix}\alpha_0\\ \alpha_1\end{Bmatrix}=\begin{bmatrix}1 & -\Delta t\\ 0 & 1\end{bmatrix}\begin{Bmatrix}a\\ \dot{a}\end{Bmatrix}_i \quad \text{and} \quad \begin{Bmatrix}-g\\ h\end{Bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}a\\ \dot{a}\end{Bmatrix}_{i-1}.$$

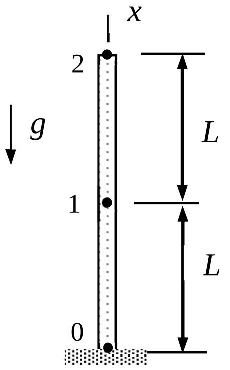
Substituting into to the linear equations given by the weighted residual method

$$\begin{bmatrix} k\frac{1}{2}\Delta t^2-m & k\frac{1}{6}\Delta t^3 \\ k\Delta t & \frac{1}{2}\Delta t^2+m \end{bmatrix}\begin{bmatrix}1 & -\Delta t\\ 0 & 1\end{bmatrix}\begin{Bmatrix}a\\ \dot{a}\end{Bmatrix}_i=m\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}a\\ \dot{a}\end{Bmatrix}_{i-1}$$

which gives after simplification the iteration

$$\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i = \frac{2}{12+\alpha^4}\begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1}, \quad \alpha = \sqrt{\frac{k}{m}}\Delta t .$$ ←

The bar shown is loaded by its own weight. Determine the displacements at the grid points 1 and 2 using the Finite Element Method. Cross-sectional area A, Young's modulus E, and density ρ of the material are constants.



Solution

The difference equations to the bar and string problems according to the Finite Element Method are given by (omitting the time derivatives and initial conditions)

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i+f'\Delta x=0 \quad i\in\{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1-a_0)+F_0+f'\frac{\Delta x}{2}=0 \text{ or } a_0=\underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1}-a_n)+F_n+f'\frac{\Delta x}{2}=0 \text{ or } a_n=\underline{a}_n,$$

where in this case of a bar problem $k=EA$, $f'=-\rho Ag$, $\Delta x=L$, and $a=u$. Equations for the three grid points are

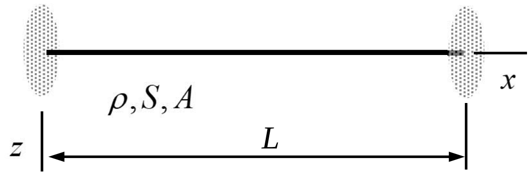
$$u_0=0, \; \frac{EA}{L}(u_0-2u_1+u_2)-\rho AgL=0, \text{ and } \frac{EA}{L}(u_1-u_2)-\rho Ag\frac{L}{2}=0.$$

In matrix notation

$$-\frac{EA}{L}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}+\frac{1}{2}\rho AgL\begin{Bmatrix}-2\\ -1\end{Bmatrix}=0 \quad \Leftrightarrow$$

$$\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\frac{1}{2}\frac{\rho gL^2}{E}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}^{-1}\begin{Bmatrix}-2\\ -1\end{Bmatrix}=\frac{1}{2}\frac{\rho gL^2}{E}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}-2\\ -1\end{Bmatrix}=-\frac{1}{2}\frac{\rho gL^2}{E}\begin{Bmatrix}3\\ 4\end{Bmatrix}. \quad \leftarrow$$

Consider the string of tightening S and mass per unit length ρA shown. Use the Finite Element Method on a regular grid $i \in \{0,1,\ldots,n\}$ to find the angular velocities ω_k of the free vibrations using the solution trial $w_i = a_k(t)\sin(k\pi i/n)$ $k \in \{1,2\ldots,n-1\}$.



Solution

The difference equations to the bar and string problems according to the Finite Element Method are given by (omitting the initial conditions as they are needed in modal analysis)

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i+f'\Delta x = m'\frac{\Delta x}{6}(\ddot{a}_{i-1}+4\ddot{a}_i+\ddot{a}_{i+1}) \quad i \in \{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1-a_0)+F_0+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_0+\ddot{a}_1)=0 \quad \text{or} \quad a_0=\underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1}-a_n)+F_n+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_n+\ddot{a}_{n-1})=0 \quad \text{or} \quad a_n=\underline{a}_n.$$

Where in this case of a bar problem $k = S$, $m' = \rho A$, $\Delta x = L/n$, $a = w$ and external forces vanish. As the trial solution satisfies the fixed end conditions, it is enough to consider a typical equation inside the domain. Substituting the trial solution and using the trigonometric identity $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$

$$w_{i-1}-2w_i+w_{i+1} = -2(1-\cos\frac{k\pi}{n})a_k(t)\sin(k\pi\frac{i}{n}),$$

$$\ddot{w}_{i-1}+4\ddot{w}_i+\ddot{w}_{i+1} = 2(2+\cos\frac{k\pi}{n})\ddot{a}_k(t)\sin(k\pi\frac{i}{n})$$

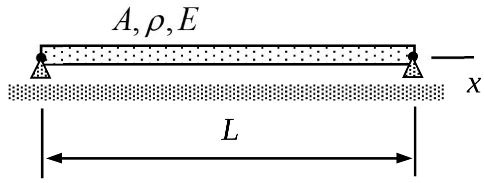
so the equation of motion simplifies to

$$\ddot{a}_k+\omega_k^2 a_k = 0 \quad \text{where} \quad \omega_k = \frac{1}{L}\sqrt{6\frac{S}{\rho A}n^2\frac{1-\cos k\pi/n}{2+\cos k\pi/n}}. \quad \leftarrow$$

In the limit, when $n \to \infty$ the angular velocity coincides with that of the continuum model

$$\omega_k = k\frac{\pi}{L}\sqrt{\frac{S}{\rho A}}.$$

A bar is free to move in the horizontal direction as shown. Write the equation system $\mathbf{Ka}+\mathbf{M\ddot{a}}=0$ given by the Finite Element Method on a regular grid with $i\in\{0,1,2\}$. Also, determine the angular velocities and modes of the free vibrations. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

The difference equations to the bar and string problems according to the Finite Element Method are given by (omitting the initial conditions as they are needed in modal analysis)

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i+f'\Delta x=m'\frac{\Delta x}{6}(\ddot{a}_{i-1}+4\ddot{a}_i+\ddot{a}_{i+1}) \qquad i\in\{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1-a_0)+F_0+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_0+\ddot{a}_1)=0 \quad \text{or} \quad a_0=\underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1}-a_n)+F_n+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_n+\ddot{a}_{n-1})=0 \quad \text{or} \quad a_n=\underline{a}_n,$$

Where in this case of a bar problem $k=EA$, $m'=\rho A, \Delta x=L$, $a=u$ and external forces vanish. Equations for the three grid points are

$$\frac{EA}{L}(u_1-u_0)-\rho A\frac{L}{6}(2\ddot{u}_0+\ddot{u}_1)=0$$

$$\frac{AE}{L}(u_0-2u_1+u_2)-\rho A\frac{L}{6}(\ddot{u}_0+4\ddot{u}_1+\ddot{u}_2)=0$$

$$\frac{EA}{L}(u_1-u_2)-\rho A\frac{L}{6}(2\ddot{u}_2+\ddot{u}_1)=0$$

In matrix notation, the equation are

$$\frac{EA}{L}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_0\\ u_1\\ u_2\end{Bmatrix}+\rho A\frac{L}{6}\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_0\\ \ddot{u}_1\\ \ddot{u}_2\end{Bmatrix}=0$$

Modal analysis uses the trial solution $\mathbf{u}=\mathbf{A}e^{\mathrm{i}\omega t}$ in which $\mathbf{A}$ represent mode and ω the corresponding angular velocity. Substitution into the set of differential equations gives a set of algebraic equations for the possible combinations $(\omega,\mathbf{A})$:

$$(\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}-\lambda\begin{bmatrix}2 & 1 & 0\\ 1 & 4 & 1\\ 0 & 1 & 2\end{bmatrix})\begin{Bmatrix}A_0\\ A_1\\ A_2\end{Bmatrix}=0 \quad \text{where} \quad \lambda=\frac{\rho}{E}\frac{L^2}{6}\omega^2 \quad \Leftrightarrow \quad \omega=\frac{1}{L}\sqrt{6\lambda\frac{E}{\rho}}.$$

First, the possible λ values:

$$\det\left(\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right) = 0 \quad \Rightarrow \quad \lambda = 0 \quad \text{or} \quad \lambda = \frac{1}{2} \quad \text{or} \quad \lambda = 2\,.$$

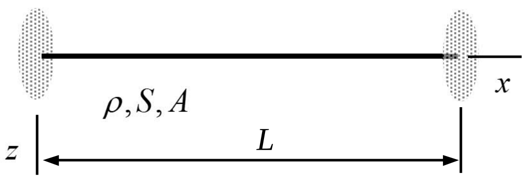
Then the corresponding modes one-by-one

$$\lambda_1 = 0: \quad \omega_1 = 0, \quad \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = 0 \quad \Rightarrow \mathbf{A}_1 = \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \quad \leftarrow$$

$$\lambda_2 = \frac{1}{2}: \quad \omega_2 = \frac{2}{L}\sqrt{3\frac{E}{\rho}}, \quad \begin{bmatrix} 0 & -3 & 0 \\ -3 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = 0 \quad \Rightarrow \mathbf{A}_2 = \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \leftarrow$$

$$\lambda_3 = 2: \quad \omega_3 = \frac{4}{L}\sqrt{3\frac{E}{\rho}}, \quad \begin{bmatrix} -3 & -3 & 0 \\ -3 & -6 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = 0 \quad \Rightarrow \mathbf{A}_3 = \begin{Bmatrix} A_0 \\ A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}. \quad \leftarrow$$

Consider the string of tightening S and mass per unit length ρA shown. First, use the Finite Element Method with the second order accurate central differences on a regular grid $i \in \{0,1,2\}$ to find the equation of motion of the form $ka + m\ddot{a} = 0$. Second, write the iteration equation for a typical time-step of size Δt according to Discontinuous Galerkin Method giving the values of displacement and velocity on the temporal grid.



Solution

The generic equation set for the model problems and the Finite Element Method on a regular grid is given by

$$\frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F_i + f'\Delta x = m'\frac{\Delta x}{6}(\ddot{a}_{i-1} + 4\ddot{a}_i + \ddot{a}_{i+1}) \quad i \in \{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1 - a_0) + F_0 + f'\frac{\Delta x}{2} - m'\frac{\Delta x}{6}(2\ddot{a}_0 + \ddot{a}_1) = 0 \quad \text{or} \quad a_0 = \underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1} - a_n) + F_n + f'\frac{\Delta x}{2} - m'\frac{\Delta x}{6}(2\ddot{a}_n + \ddot{a}_{n-1}) = 0 \quad \text{or} \quad a_n = \underline{a}_n,$$

$$a_i - g_i = 0 \quad \text{and} \quad \dot{a}_i - h_i = 0.$$

In the string application of the problem, external forces vanish, $k = S$, $m' = \rho A$, $a = w$, $\Delta x = L/2$., and external forces vanish. Equations for $i \in \{0,1,2\}$ simplify to

$$w_0 = 0, \quad 2\frac{S}{L}(w_0 - 2w_1 + w_2) = \rho A\frac{L}{12}(\ddot{w}_0 + 4\ddot{w}_1 + \ddot{w}_2), \text{ and } w_2 = 0 \quad t > 0,$$

$$w_1 = g_1 \quad \text{and} \quad \dot{w}_1 = h_1 \quad t = 0.$$

In solution methods for time dependent problem, algebraic equations are used to eliminate the displacements of the boundary points from the differential equation, so the initial value problem simplifies to

$$4\frac{S}{L}w_1 + \rho A\frac{L}{12}4\ddot{w}_1 = 0 \quad t > 0, \quad w_1 = g \quad \text{and} \quad \dot{w}_1 = h \quad t = 0.$$

With definition $w_1 = a$, time integration by Discontinuous-Galerkin method is given by iteration

$$\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i = \frac{2}{12+\alpha^4}\begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix}_0 \quad \text{where } \alpha = \sqrt{12\frac{S}{\rho A}}\frac{\Delta t}{L}.$$