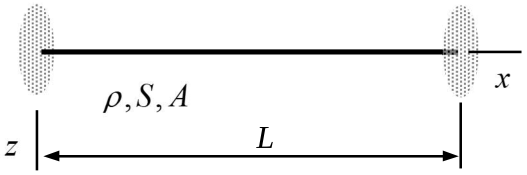
Name________________________ Student number________________

Home assignment 2

What is relative (numerical) error in the smallest frequency of the free vibrations $f_{\min}$ of the string shown if predicted by the Finite Difference Method and a regular spatial grid $i \in \{0,1,2,3\}$? Cross-sectional area A, density of the material ρ, and horizontal tightening S are constants. Use the three-point central difference approximation to the second derivative with respect to x. The smallest frequency given by the continuum model $f_{\min} = \sqrt{S/(\rho A)}/(2L)$.



Solution

The equations for the boundary points and for the two points inside the domain, as given by the 2:nd order accurate central difference approximation to the second derivative (with respect to *x*), are

$$w_0 = 0 \;,\; \frac{S}{\Delta x^2}(w_0 - 2w_1 + w_2) = \rho A \ddot{w}_1 \;,\; \frac{S}{\Delta x^2}(w_1 - 2w_2 + w_3) = \rho A \ddot{w}_2 \text{, and } \; w_3 = 0 \,.$$

In matrix notation and $\Delta x = L/3$, the equations for points 1 and 2 are (when the known displacements at the boundary points are used there)

$$\frac{S}{\Delta x^2}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + \rho A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix} = 0 \,. \quad \leftarrow$$

Frequencies and modes of the free vibrations follow with the trial solution $\mathbf{a} = \mathbf{A}\exp(\mathrm{i}\omega t)$. Using the notation $\lambda = \omega^2 \Delta x^2 \rho A / S$, the conditions for the possible angular velocity ω and mode $\mathbf{A}$ pairs takes the form

$$(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \;.$$

A homogeneous linear equation system can yield a non-zero solution only if the matrix is singular. The condition implies that

$$\det\begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 - 1 = 0 \quad \Rightarrow \quad \lambda_1 = 1 \;\text{ or }\; \lambda_2 = 3 \,.$$

The angular velocities follow from the known relationship $\lambda = \omega^2 \Delta x^2 \rho A / S$ and frequency from $\omega = 2\pi f$. The smallest frequency is given by $\lambda_1 = 1$:

$\underline{f}_{\min} = \frac{3}{2\pi}\frac{1}{L}\sqrt{\frac{S}{\rho A}}$ the exact value being $f_{\min} = \frac{1}{2}\frac{1}{L}\sqrt{\frac{S}{\rho A}}$.

The relative error in the smallest value $\frac{\underline{f}_{\min} - f_{\min}}{f_{\min}} 100\% = (\frac{3}{\pi} - 1)100\% \approx -5\%$.