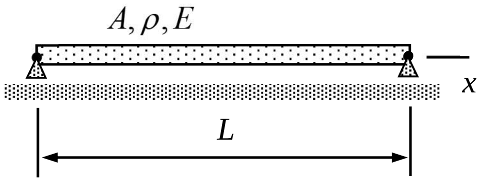
Name_________________________ Student number__________________

Home assignment 3

A bar is free to move in the horizontal direction as shown. At $t=0$, the bar moves with constant velocity $\dot{U}$ to the direction of the $x-$axis displacements being zeros. Use the Finite Difference Method on a regular grid with $i \in \{0,1,2\}$ and the Crank-Nicolson method with step size Δt to find the displacements and velocities at $t=\Delta t$. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

The continuum model for the problem is given by equations

$$EA\frac{\partial^2 u}{\partial x^2}=\rho A\frac{\partial^2 u}{\partial t^2} \quad x\in]0,L[\quad \text{and} \quad n_x EA\frac{\partial u}{\partial x}=0 \quad x\in\{0,L\} \quad t>0,$$

$$\frac{\partial u}{\partial t}=\dot{U} \quad \text{and} \quad u=0 \quad x\in]0,L[\quad t=0.$$

In the Finite Difference Method, derivatives with respect to the spatial coordinates are replaced by difference approximations. Assuming the central three-point approximation to the second derivative and two point backward and forward approximations to the derivatives in the boundary conditions (difference approximations cannot use points outside the region), the equations for the points $i\in\{0,1,2\}$ become

$$\frac{EA}{\Delta x}(u_0-u_1)=0, \quad \frac{EA}{\Delta x^2}(u_0-2u_1+u_2)=\rho A\ddot{u}_1, \quad \frac{EA}{\Delta x}(u_2-u_1)=0 \quad t>0,$$

$$\dot{u}_1=\dot{U} \quad \text{and} \quad u_1=0 \quad t=0$$

of which the first and the last are algebraic equations and do not require discretization in the temporal domain. Also, the initial conditions apply only to the points inside the domain, i.e., to point 1 whose equation is an ordinary second order differential equation in time. The algebraic equations can be used to eliminate u_2 and u_0 from the differential equation to get the initial value problem

$$\rho A\ddot{u}_1=0 \quad t>0, \quad \dot{u}_1=\dot{U} \quad t=0, \text{ and } \quad u_1=0 \quad t=0$$

having the exact solution $u_1(t)=\dot{U}t$ and using the algebraic equations of the boundary conditions $u_0(t)=u_2(t)=u_1(t)=\dot{U}t$. Let us apply the Crank-Nicolson method to see how well it predicts the displacements:

$$\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_i = \frac{1}{4+\alpha^2}\begin{bmatrix} 4-\alpha^2 & 4 \\ -4\alpha^2 & 4-\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_{i-1} \quad \text{and} \quad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta th \end{Bmatrix}.$$

With the present problem $a=u_1$ and $\alpha=\sqrt{k/m}\Delta t=0$ so the iteration simplifies to

$$\begin{Bmatrix} u_1 \\ \Delta t\dot{u}_1 \end{Bmatrix}_i = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ \Delta t\dot{u}_1 \end{Bmatrix}_{i-1} \quad \text{and} \quad \begin{Bmatrix} u_1 \\ \Delta t\dot{u}_1 \end{Bmatrix}_0 = \begin{Bmatrix} 0 \\ \Delta t\dot{U} \end{Bmatrix}$$

which gives after one step

$$\begin{Bmatrix} u_1 \\ \Delta t\dot{u}_1 \end{Bmatrix}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ \Delta t\dot{u}_1 \end{Bmatrix}_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \Delta t\dot{U} \end{Bmatrix} = \begin{Bmatrix} \Delta t\dot{U} \\ \Delta t\dot{U} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u_1 \\ \dot{u}_1 \end{Bmatrix}_1 = \begin{Bmatrix} \Delta t\dot{U} \\ \dot{U} \end{Bmatrix}.$$

The displacements and velocities of the boundary points follow from the two algebraic equations given by the boundary conditions in the same manner as with the exact solution with respect to time:

$$\begin{Bmatrix} u_0 \\ u_1 \\ u_2 \end{Bmatrix}_1 = \dot{U}\Delta t\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} \dot{u}_0 \\ \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}_1 = \dot{U}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}. \leftarrow$$

In this particular case, the outcome by the numerical time integration method coincides with the exact solution.