Mathematics for economists: ECON-C1000
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Economic applications of constrained optimization

Utility maximization problem (UMP)

A consumer allocates her budget of $w > 0$ to n goods. Her consumption vector is an element of the positive orthant of the n Euclidean space $X = \{\boldsymbol{x} \in \mathbb{R}^n_+\}$. We assume that the consumer has a continuous utility function $u(x)$ defined on X. Economic scarcity is present through the budget constraint:

$$\boldsymbol{p} \cdot \boldsymbol{x} \leq w \text{ or } \sum_{i=1}^{n} p_i x_i \leq w,$$

where $\boldsymbol{p} = (p_1, ..., p_n) > 0$ is the vector of strictly positive prices for the goods.

We can write this problem then as
Maximize

$$u(x_1, ..., x_n)$$

subject to

$$\sum_{i=1}^{n} p_i x_i \leq w,$$

$$x_i \geq 0 \text{ for all } i.$$

By writing the constraints in the equivalent form:

subject to

$$\sum_{i=1}^{n} p_i x_i - w \leq 0,$$
$$-x_i \leq 0 \text{ for all } i,$$

the problem is in the standard form that we always write for inequality constrained optimization problems.

Let's pause to see what we know about this problem already. To see that the feasible set is bounded, let $p^{min} = \min_j p_j$ (i.e. one of the smallest prices p_j). Then we know that for all feasible x, we have $p_i x_i \leq w$ for all i since $x_i \geq 0$ and $p_i > 0$ for all i. Therefore for all feasible $\boldsymbol{x}$, $x_i \leq \frac{w}{p^{min}}$ for all i. In other words, the feasible set is bounded since $0 \leq x_i \leq \frac{w}{p^{min}}$ for all i.

To see that the feasible set is closed, we need to show that all limit points of the feasible belong to the feasible set. We show this by arguing that when $\boldsymbol{y}$ is not in the feasible set, it is not a limit point. If $\boldsymbol{y}$ is not feasible, then either $y_i < 0$ for some i or $\sum_i p_i y_i > w$. In the first case, if $y_i = -a$ for some $a > 0$, then no $B^\varepsilon(\boldsymbol{y})$ contains any feasible point for $\varepsilon < a$. Hence y is not a limit point.

For the second case $\sum_i p_i y_i - w = b$ for some $b > 0$, let $p^{max} = \max_j p_j$. Then no point in $B^\varepsilon(y)$ for $\varepsilon < \frac{b}{p^{max}}$ is feasible. Hence $\boldsymbol{y}$ is not a limit point.

Remark 1. You do not have to prove this closedness property every time. It is sufficient to note that $\{\boldsymbol{x} \in \mathbb{R}^n : g(\boldsymbol{x}) \leq 0\}$ is a closed set whenever g is continuous. In our case here, all g_k are linear and therefore continuous. For multiple constraints, just observe that intersections of closed sets are closed.

Hence we know by Weierstrass' theorem that a maximum exists. Since the constraint functions are linear, the feasible set is convex. If u is strictly increasing (as we usually assume) and quasiconvex, then the first order Kuhn-Tucker conditions are necessary and sufficient for optimum. In words, whenever we find a point satisfying the K-T conditions, we have solved the problem.

Let's turn our attention next to the Lagrangean and the K-T conditions:

$$\mathcal{L}(\boldsymbol{x}, \lambda) = u(\boldsymbol{x}) - \lambda_0 \left[\sum_{i=1}^{n} p_i x_i - w\right] + \sum_{i=1}^{n} \lambda_i x_i$$

The first-order K-T conditions are:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}} = \frac{\partial u(\boldsymbol{x})}{\partial x_i} - \lambda_0 p_i + \lambda_i = 0 \text{ for all } i, \tag{1}$$

$$\lambda_0 \left[\sum_{i=1}^{n} p_i x_i - w\right] = 0, \tag{2}$$

$$\lambda_i x_i = 0 \text{ for all } i, \tag{3}$$

$$\sum_{i=1}^{n} p_i x_i - w \leq 0, \tag{4}$$

$$-x_i \leq 0 \text{ for all } i, \tag{5}$$

$$\lambda_i \geq 0 \;\; i \in \{0, 1, ..., n\}. \tag{6}$$

If the utility function has a strictly positive partial derivative for some x_i at the optimum, then the budget constraint must bind and $\lambda_0 > 0$. This follows immediately from the first line of the K-T conditions. For the other inequality constraints, consider the partial derivatives at $\boldsymbol{x} \in X$ with $x_i \to 0$ for some i. If

$$\lim_{x_i \to 0} \frac{\partial u}{\partial x_i} = \infty,$$

then we know again from the first line of the K-T conditions that at optimum $x_i > 0$. (Intuitively, if you have an infinitely large marginal utility for some good, you would want to consume more of it). If this is true for all i, then we can ignore the non-negativity constraints and we are effectively back to a problem with a single equality constraint.

If $\frac{\partial u(\boldsymbol{x})}{\partial x_i} < \infty$ for $\boldsymbol{x} = (x_i, \boldsymbol{x}_{-i}) = (0, \boldsymbol{x}_{-i})$, then we must also consider corner solutions where $x_i = 0$ at optimum. For interior solutions, we get from the first equation by eliminating λ the familiar condition:

$$\frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_k}} = \frac{p_i}{p_k}. \tag{7}$$

This is of course the familiar requirement that $MRS_{x_i,x_k} = \frac{p_i}{p_k}$ that we saw in Principles of Economics 1. Now we see that the same condition extends for many goods and the economic intuition is exactly the same. The price ration gives the marginal rate of transformation between the different goods and at an interior optimum, that rate must coincide with the marginal rate of substitution. In some cases, the functional form is such that the problem can be solved explicitly.

Constant elasticity of substitution

We start with the functional form that we have already seen a number of times in this course:

$$u(\boldsymbol{x}) = (a_1 x_1^\rho + \cdots + a_n x_n^\rho)^{\frac{1}{\rho}},$$

for $\rho > 1$. You have already shown in problem sets that functions of this type are quasiconcave. We compute the marginal utility for each x_i:

$$\frac{\partial u}{\partial x_i} = \rho a_1 x_i^{\rho-1} \frac{1}{\rho} (a_1 x_1^\rho + \cdots + a_n x_n^\rho)^{\frac{1}{\rho}1}.$$

Note that since $\rho < 1$, we have $\frac{\partial u}{\partial x_i} > 0$, and

$$\lim_{x_i \to 0} \frac{\partial u}{\partial x_i} = \infty.$$

Since the feasible set is convex and the objective function is quasiconvcave with a non-vanishing derivative, the first order conditions are necessary and sufficient for optimum. Since the marginal utility is unbounded at the boundary, we know that we have an interior solution and that the budget constraint is binding. Hence the K-T conditions require simply that for all i, k:

$$\frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_k}} = \frac{p_i}{p_k},$$

and the budget constraint holds as an equality:

$$\sum_{i=1}^{n} p_i x_i = 0.$$

Hence we have that

$$\frac{a_1 x_1^{\rho-1}}{a_k x_k^{\rho-1}} = \frac{p_1}{p_k},$$

or

$$\frac{x_1}{x_k} = (\frac{a_k p_1}{a_1 p_k})^{\frac{1}{\rho-1}},$$

or

$$x_k = x_1(\frac{a_k p_1}{a_1 p_k})^{\frac{1}{1-\rho}}. \tag{8}$$

Substituting into the budget constraint, we get:

$$p_1 x_1 + \sum_{k=2}^{n} p_k x_1(\frac{a_k p_1}{a_1 p_k})^{\frac{1}{1-\rho}} = w.$$

We can solve for x_1 to get

$$x_1 = \frac{w}{p_1 + \sum_{k=2}^{n} p_k(\frac{a_k p_1}{a_1 p_k})^{\frac{1}{1-\rho}}}.$$

Substituting into (8), we can solve the other x_j.

To get a bit nicer expression, let $r = \frac{\rho}{\rho-1}$ and assume that $a_i = 1$ for all i. Then we have for each j:

$$x_j = \frac{w p_j^{r-1}}{\sum_{k=1}^{n} p_k^r}.$$

(Exercise: Check that you can get this formula from the equation above).

In this case, we are able to solve the optimal demands as explicit functions of the exogenous variables. We call the optimal solutions to the utility maximization problem the Marshallian demands. You will see these demand functions for CES utility functions in your future studies in models of monopolistic competition as needed in growth theory, international trade and industrial organization.

If you want to understand where the name constant elasticity of substitution comes from, you should note that

$$\frac{x_i}{x_k} = (\frac{p_i}{p_k})^{\frac{1}{\rho-1}}(\frac{a_k}{a_i})^{\frac{1}{\rho-1}}.$$

Hence a small percentage change in the price ratio between any two goods induces the same percentage change in the optimal consumptions. The size of this change is given by $\frac{1}{\rho-1}$ and hence ρ measures the elasticity of substitution between any two goods. The higher, ρ, the higher the substitution away from a good when its price increases.

You should consider the comparative statics of the optimal demands in prices and income. In other words, compute the partial derivatives $\frac{\partial x_i(\boldsymbol{p},w)}{\partial p_i}$, $\frac{\partial x_i(\boldsymbol{p},w)}{\partial p_j}$ and $\frac{\partial x_i(\boldsymbol{p},w)}{\partial w}$. For example, when does the demand for good i increase in the price of another good p_j?

Let's look at some special cases. In Problem set 1, you showed that as $\rho \to 0$, the CES -function converges to the Cobb-Douglas utility function $u(\boldsymbol{x}) = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

If we just substitute $\rho = 0$ into the optimal demand, we get

$$x_i = \frac{\alpha_i w}{p_i(\sum_{i=1}^n \alpha_i)}.$$

For the Cobb-Douglas utility function, you get the result that the expenditure share $\frac{p_i x_i}{w}$ on each good is equal to $\frac{\alpha_i}{(\sum_{i=1}^n \alpha_i)}$. In this case, the consumer's expenditure share does not depend on her wealth. In other words, rich and poor consumers use the same fraction of their income on food, clothing, yachts etc. This is clearly not a very good description of reality.

By equation (8), you can see that CES -functions do not offer that much help either. The expenditure shares are still constant in wealth (even though they depend now on the entire price vector). One way to get more realistic consumption patters is to define the utility function for consumptions above a level needed for subsistence. Let $\underline{\boldsymbol{x}} = (\underline{x_1}, ..., \underline{x_n})$ be the levels of each good needed for survival and assume that $w \geq \boldsymbol{p} \cdot \underline{\boldsymbol{x}}$. The utility function for $\boldsymbol{x} \in \mathbb{R}^n$ such that $x_i \geq \underline{x_i}$ is of Cobb-Douglas -like form:

$$u(\boldsymbol{x}) = (x_1 - \underline{x_1})^{\alpha_1} \ldots (x_n - \underline{x_n})^{\alpha_n},$$

where $0 < \alpha_i < 1$ for all i and $\sum_{i=1}^n \alpha_i = 1$. Notice that the marginal utility for good i is infinite if $x_i = \underline{x_i}$ and that the utility function is strictly increasing in all of its components. Hence we still have an interior solution and the budget constraint binds.

We get as above:

$$\frac{\frac{\partial u(\boldsymbol{x})}{\partial x_i}}{\frac{\partial u(\boldsymbol{x})}{\partial x_k}} = \frac{\alpha_i(x_i - \underline{x_i})}{\alpha_k(x_k - \underline{x_k})} = \frac{p_i}{p_k} \text{ for all } i, k,$$

and

$$\sum_{i=1}^n p_i x_i = w.$$

We get that

$$x_i - \underline{x_i} = \frac{\alpha_i p_1}{\alpha_1 p_i}(x_1 - \underline{x_1}) \text{ for all } i. \tag{9}$$

Multiplying both sides by p_i and summing over i gives:

$$\sum_{i=1}^n p_i(x_i - \underline{x_i}) = \frac{p_1 \sum_{i=1}^n \alpha_i}{\alpha_1}(x_1 - \underline{x_1}).$$

So we can solve:

$$x_1 - \underline{x_1} = \frac{\alpha_1(w - \sum_{i=1}^n p_i \underline{x_i})}{p_1},$$

where we used the budget constraint $\sum_{i=1}^n p_i x_i = w$ and $\sum_{i=1}^n \alpha_i = 1$

By (9), we see that

$$x_k - \underline{x_k} = \frac{\alpha_k(w - \sum_{i=1}^{n} p_i \underline{x_i})}{p_k}.$$

Now you can see that the consumer uses a constant fraction of her excess income (above what is needed for the necessities $\underline{\boldsymbol{x}}$) in constant shares given by the α_i. Since the poor have less excess wealth, their consumption fractions are closer to the ones given by the subsistence levels $\beta_i := \frac{\underline{x_i}}{\sum_i \underline{x_i}}$. Hence the richest spend fractions α_i 0n good i and the poorest spend β_i.

Expenditure minimization problem

We cover briefly the related problem of minimizing expenditure subject to the constraint of reaching a specified level of utility. All the notation is exactly as in the previous subsection. We assume that the utility function that we have is quasiconcave.

$$\min_{\boldsymbol{x} \in X} \boldsymbol{p} \cdot \boldsymbol{x} = \sum_{i=1}^{n} p_i x_i,$$

subject to

$$u(\boldsymbol{x}) \geq \overline{u}.$$

This means that we have a linear and thus quasiconvex objective function for our minimization problem and since the utility function is quasiconcave, the feasible set is convex. Hence we know that K-T necessary conditions are also sufficient. Notice that the feasible set is now not bounded (why?), but the solution exists because we can take any $\boldsymbol{x}*$ such that $u(x*) \geq \overline{u}$ and restrict attention to $\boldsymbol{x}$ such that

$$\boldsymbol{p} \cdot \boldsymbol{x} \leq \boldsymbol{p} \cdot \boldsymbol{x}*,$$

since $\boldsymbol{x}*$ is a feasible solution. But this set is convex and bounded since it is a budget set.

The Lagrangean to the problem is:

$$\mathcal{L}(\boldsymbol{x},\lambda)=\sum_{i=1}^{n}p_i x_i-\lambda_0(\overline{u}-u(\boldsymbol{x}))+\sum_{i=1}^{n}\lambda_i x_i.$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{x}}=p_i+\lambda_0\frac{\partial u}{\partial x_i}+\lambda_i=0 \text{ for all } i, \tag{10}$$

$$\lambda_0[u(x)-\overline{u}]=0, \tag{11}$$

$$\lambda_i x_i=0 \text{ for all } i, \tag{12}$$

$$\overline{u}-u(x)\leq 0, \tag{13}$$

$$-x_i\leq 0 \text{ for all } i, \tag{14}$$

$$\lambda_i\geq 0 \;\; i\in\{0,1,...,n\}. \tag{15}$$

Notice that for interior solutions (where $\lambda_1=\lambda_2=...=\lambda_n=0$, we get again (after eliminating the multiplier) from the first line of the K-T conditions that

$$\frac{\frac{\partial u(\boldsymbol{x})}{\partial x_i}}{\frac{\partial u(\boldsymbol{x})}{\partial x_k}}=\frac{p_i}{p_k}.$$

We have exactly the same situation as before. Now the ratio of marginal utilities is really the MRT for the problem since it describes the feasible set. The price ratio is now the MRS of this new problem. We will relate these two problems in the next lecture.

Cost minimization problem for a firm

A firm chooses its inputs k,l to minimize the cost of reaching a production target of $\overline{q}$ at given input prices r,w. The production function is assumed to be a strictly increasing and quasiconcave function $f(k,l)$.

$$\min_{(k,l)\in\mathbb{R}^2_+} rk+wl$$

subject to

$$f(k, l) \geq \overline{q}.$$

Notice that this is the same mathematical problem as in expenditure minimization. Only the names of variables have changed. The solution to the problem is therefore also identical and we do not repeat it here.