Mathematics for Economists: Lecture 9

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This lecture covers

1. Economic applications of constrained optimization
   1.1 Utility maximization continued
   1.2 Expenditure and cost minimization
   1.3 Portfolio choice
2. First look at duality and value functions

Quasilinear utility function

- We end the section on utility maximization with $u(x, y) = v(x) + y$, where v is a strictly increasing and strictly concave function subject to non-negativity of x, y and the budget constraint

$$p_x x + y \leq w.$$

- Are we losing generality in assuming that $p_y = 1$?
- Now $MRS_{x,y} = v'(x)$.

Quasilinear utility function

- If $v'(\frac{w}{p_x}) > p_x$, or if $v'(0) < p_x$, then we have a corner solution.
- In the first case, $x(p_x, w) = \frac{w}{p_x}$, $y(p_x, w) = 0$.
- In the second case, $x(p_x, w) = 0$ and $y(p_x, w) = w$.
- Otherwise $x(p_x, w)$ solves

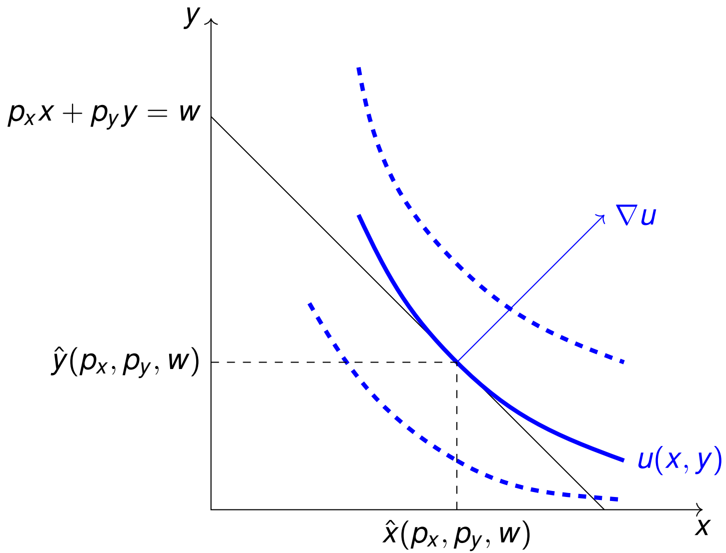
$$v'(x) = p_x,$$

  and

$$y = (w - p_x x(p_x, w)).$$

- Notice that $MRS_{x,y}$ does not depend on y. A higher y simply shifts vertically the indifference curves.
- This utility function lies behind partial equilibrium analysis in microeconomics where x is sold in the market of interest and y is everything else.
- y represents expenditure on all other goods or total income. With quasi-linear utility, there are no income effects (as long as we remain in the range for interior solutions).

Figure: Utility maximization problem



Expenditure minimization problem

- Suppose the consumer has a utility function given by $u(x, y)$
- How do you minimize expenditure to reach at least the utility level $\bar{u}$?

$$\min_{x,y} p_x x + p_y y$$

subject to:

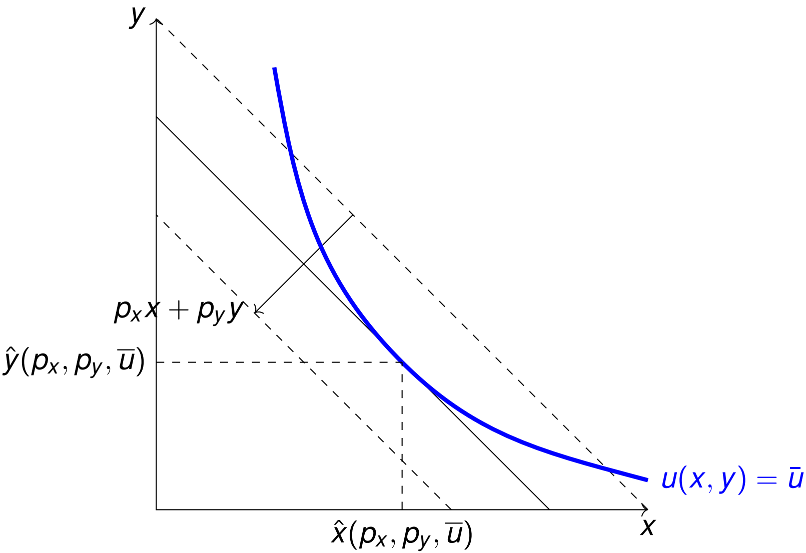
$$x, y \geq 0, \quad u(x, y) \geq \bar{u}.$$

- We'll connect UMP and EMP in the second part of this lecture.
- Lagrangean for the problem:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3) = p_x x + p_y y - \lambda_1 (u(x, y) - \bar{u}) + \lambda_2 x + \lambda_3 y.$$

- Exercise: What are the Kuhn-Tucker first-order conditions for this problem?

Figure: Expenditure minimization problem



Cost minimization problem for a firm

- A firm chooses its inputs k, l to minimize the cost of reaching a production target of $\overline{q}$ at given input prices r, w.
- Notice that the objective function is quasiconvex.
- The production function is assumed to be a strictly increasing and quasiconcave function $f(k, l)$.

$$\min_{(k,l)\in\mathbb{R}^2_+} rk + wl$$

subject to

$$f(k, l) \geq \bar{q}, k, l \geq 0.$$

Cost minimization problem for a firm

- Notice that the feasible set is closed and convex.
- It is not bounded but is this a problem for existence of a solution?
- Lagrangean for the problem:

$$\mathcal{L}(k, l, \lambda_1, \lambda_2, \lambda_3) = rk + wl - \lambda_1(f(k, l) - \bar{q}) - \lambda_2 k - \lambda_3 l.$$

- It is often assumed that $f(0, l) = f(k, 0) = 0$ and then the non-negativity constraints are not binding. (Of course, with e.g. linear technologies, you must consider corner solutions).

Cost minimization problem for a firm: Cobb-Douglas case

- Let $f(k, l) = k^{\alpha} l^{1-\alpha}$.
- $(\hat{k}, \hat{l})$ such that $\hat{k} = 0$ or $\hat{l} = 0$ are not in the feasible set.
- I leave it as an exercise to argue that the constraint $f(k, l) \geq \bar{q}$ binds at optimum, i.e.
$$f(\hat{k}, \hat{l}) = \bar{q}.$$
- First-order conditions for optimum are:
$$r = \lambda_1 \alpha \left(\frac{\hat{l}}{\hat{k}} \right)^{1-\alpha}, w = \lambda_1 (1 - \alpha) \left(\frac{\hat{k}}{\hat{l}} \right)^{\alpha},$$
and
$$f(\hat{k}, \hat{l}) = \bar{q}.$$

Cost minimization problem for a firm: Cobb-Douglas case

- From the first two, you get:

$$\frac{r}{w} = \frac{\alpha}{1-\alpha}\frac{\hat{l}}{\hat{k}}.$$

- Solving for $\hat{k}$ and substituting into the constraint gives:

$$\hat{k} = \bar{q}\left(\frac{\alpha w}{(1-\alpha)r}\right)^{1-\alpha}, \hat{l} = \bar{q}\left(\frac{(1-\alpha)r}{\alpha w}\right)^{\alpha}.$$

- You can also verify that $\hat{\lambda}_1 > 0$.
- The minimal cost for achieving production level $\bar{q}$ is

$$c(\bar{q}; r, w) = r\hat{k} + w\hat{l} = \bar{q}(\alpha)^{-\alpha}(1-\alpha)^{\alpha-1}r^{\alpha}w^{1-\alpha}.$$

Comparative statics of utility maximization

- Recall from Principles of Economics I,
    1. Substitution effect of price changes
    2. Income effect of price changes
- We will see how to express these mathematically by connecting utility maximization and expenditure minimization problems.
- In order to be able to do this, we need to understand the value functions of the two problems.

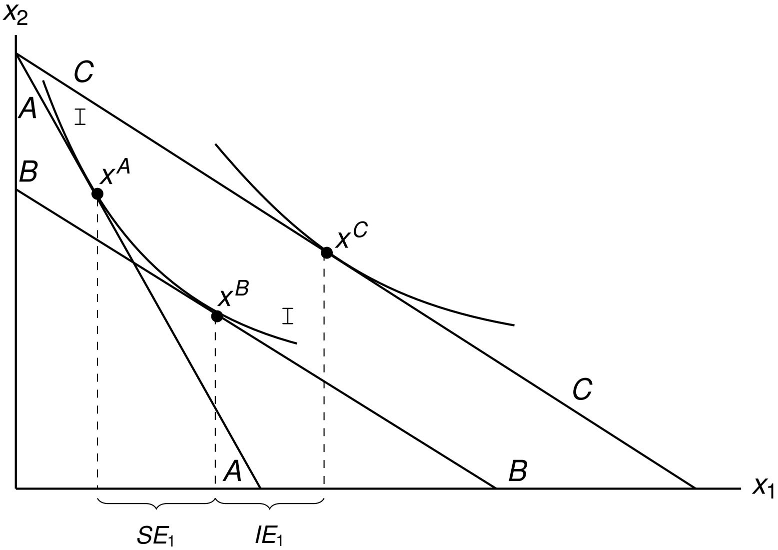


Figure: Hicks decomposition

Value function: utility maximization

- What is the highest utility level that a consumer can reach when maximizing her utility subject to a budget constraint?
- If $(x_1(p_1, ..., p_n, w), ..., x_n(p_1, ..., p_n, w))$ is her optimal demand, we get the utility level by plugging the demand back into the utility function:

$$u(x_1(p_1, ..., p_n, w), ..., x_n(p_1, ..., p_n, w)).$$

- Notice that this maximized value is a function of the exogenous variables $(\boldsymbol{p}, w)$. We call it the value function of the problem.
- For utility maximization problems, the value function is called the *indirect utility function*:

$$v(p_1, ..., p_n, w) := u(x_1(p_1, ..., p_n, w), ..., x_n(p_1, ..., p_n, w)).$$

Value function: expenditure minimization

- Let's return to the expenditure minimization problem:

$$\min_{\boldsymbol{h} \in X} \boldsymbol{p} \cdot \boldsymbol{h} = \sum_{i=1}^{n} p_i h_i,$$

  subject to

$$u(\boldsymbol{h}) = \overline{u}.$$

- Denote the solution to this problem by $\boldsymbol{h}(\boldsymbol{p}, \overline{u})$. We call $h_i(\boldsymbol{p}, \overline{u})$ the Hicksian or compensated demand for good i.
- The *value function* of this problem is the minimal expenditure needed to achieve utility level $\bar{u}$:

$$e(\boldsymbol{p}, \overline{u}) = \sum_{i=1}^{n} p_i h_i(\boldsymbol{p}, \overline{u}).$$

Connecting expenditure minimization and UMP

- Hold prices $\hat{\boldsymbol{p}}$ fixed for a moment and ask how high utility you can achieve with income w. The answer is given by the indirect utility function $v(\hat{\boldsymbol{p}}, w)$.
- Ask next what is the minimum expenditure that you must use to achieve utility $v(\hat{\boldsymbol{p}}, w)$. The following figures should convince you that for all $\hat{\boldsymbol{p}}$,

$$e(\hat{\boldsymbol{p}}, v(\hat{\boldsymbol{p}}, w)) = w.$$

- It costs you $e(\hat{\boldsymbol{p}}, \overline{u})$ to reach utility $\overline{u}$. If your budget is $e(\hat{\boldsymbol{p}}, \overline{u})$, then the maximal utility that you can reach is for all $\hat{\boldsymbol{p}}$,

$$\overline{u} = v(\hat{\boldsymbol{p}}, e(\hat{\boldsymbol{p}}, \overline{u})).$$

Figure: UMP for $w = e(p, v(p, w))$

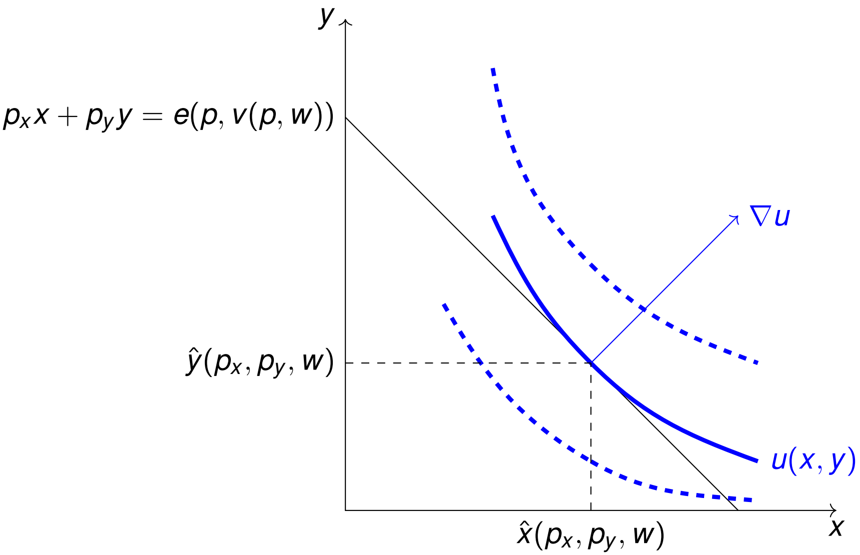
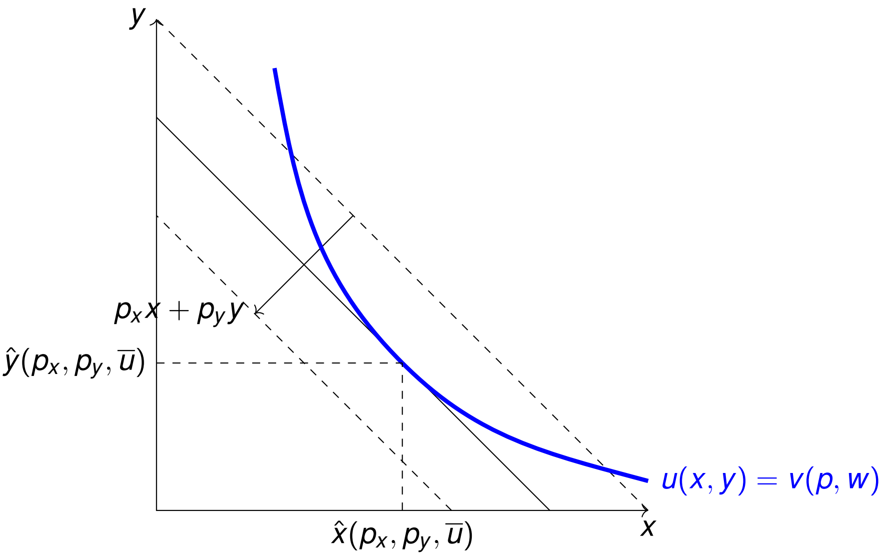


Figure: Expenditure minimization for $\overline{u} = v(p, w)$



Connecting expenditure minimization and UMP

- You can also see that for $\overline{u} = v(\boldsymbol{p}, e(\boldsymbol{p}, \overline{u})$ and $e(\boldsymbol{p}, v(\boldsymbol{p}, w)) = w$ the solutions to expenditure minimization and UMP coincide for all $\boldsymbol{p}$:

$$h_i(\boldsymbol{p}, \overline{u}) = x_i(\boldsymbol{p}, e(\boldsymbol{p}, \overline{u})) \text{ for all } i,$$

$$h_i(\boldsymbol{p}, v(\boldsymbol{p}, w)) = x_i(\boldsymbol{p}, w) \text{ for all } i.$$