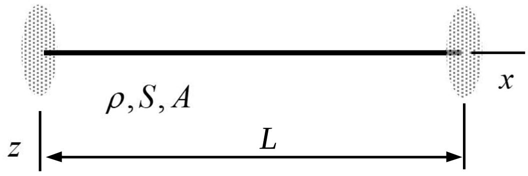
Name_____________________ Student number______________

Home assignment 2

What is relative (numerical) error in the smallest frequency of the free vibrations $\underline{f}_{\min}$ of the string shown obtained with the Finite Element Method and a regular spatial grid $i \in \{0,1,2,3\}$? Cross-sectional area A, density of the material ρ, and horizontal tightening S are constants. The smallest frequency given by the continuum model $f_{\min} = \sqrt{S/(\rho A)}/(2L)$.



Solution

The generic equation set for the string and bar models is given the Finite Element Method on a regular grid is

$$\frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F_i + f'\Delta x = m'\frac{\Delta x}{6}(\ddot{a}_{i-1} + 4\ddot{a}_i + \ddot{a}_{i+1}) \qquad i \in \{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(w_1 - w_0) + F_0 + \frac{\Delta x}{2} f' - m'\frac{\Delta x}{6}(2\ddot{a}_0 + \ddot{a}_1) = 0 \quad \text{or} \quad a_0 = \underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1} - a_n) + F_n + \frac{\Delta x}{2} f' - m'\frac{\Delta x}{6}(2\ddot{a}_n + \ddot{a}_{n-1}) = 0 \quad \text{or} \quad a_n = \underline{a}_n,$$

$$a_i - g_i = 0 \quad \text{and} \quad \dot{a}_i - h_i = 0.$$

In modal analysis, initial conditions are not needed. With $a_i = w_i$, $k = S$, and $m' = \rho A$, the equations for the grid points $i \in \{0,1,2,3\}$ simplify to

$$w_0 = 0, \quad \frac{S}{\Delta x}(w_0 - 2w_1 + w_2) - \rho A\frac{\Delta x}{6}(\ddot{w}_0 + 4\ddot{w}_1 + \ddot{w}_2) = 0,$$

$$\frac{S}{\Delta x}(w_1 - 2w_2 + w_3) - \rho A\frac{\Delta x}{6}(\ddot{w}_1 + 4\ddot{w}_2 + \ddot{w}_3) = 0, \quad w_3 = 0$$

In matrix notation, the equations for points 1 and 2 are (when the known displacements at the boundary points are used there)

$$\frac{S}{\Delta x}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} + \rho A \Delta x \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix} = 0. \quad \leftarrow$$

With the trial solution $\mathbf{a} = \mathbf{A}\exp(\mathrm{i}\omega t)$, the condition for the possible angular velocity ω and mode $\mathbf{A}$ pairs takes the form

$$(\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{where} \quad \lambda = \omega^2 \frac{1}{6}\frac{\rho A}{S}\Delta x^2,$$

A homogeneous linear equation system can give a non-zero solution only if the matrix is singular so

$$\det\begin{bmatrix} 2-4\lambda & -1-\lambda \\ -1-\lambda & 2-4\lambda \end{bmatrix} = (2-4\lambda)^2 - (1+\lambda)^2 = 0 \quad \Rightarrow \quad \lambda_1 = \frac{1}{5} \quad \text{or} \quad \lambda_2 = 1.$$

The angular velocities follow from the known relationships

$$\lambda = \omega^2 \frac{1}{6}\frac{\rho A}{S}\Delta x^2 \quad \text{and} \quad \omega = 2\pi f .$$

The smallest frequency corresponds $\lambda_{\min} = 1/5$

$$\underline{f}_{\min} = \frac{3}{2\pi}\sqrt{\frac{6}{5}}\frac{1}{L}\sqrt{\frac{S}{\rho A}} \text{ the exact being } f_{\min} = \frac{1}{2}\frac{1}{L}\sqrt{\frac{S}{\rho A}} .$$

The relative error in the smallest value

$$\frac{\underline{f}_{\min} - f_{\min}}{f_{\min}} 100\% = (\frac{3}{\pi}\sqrt{\frac{6}{5}} - 1)100\% \approx 5\% .$$