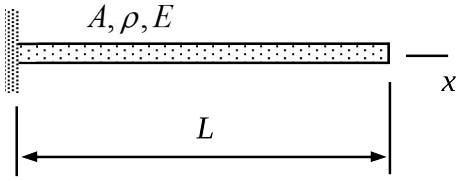
Name________________________ Student number________________

Home assignment 3

A bar is free to move in the horizontal direction as shown. At $t=0$, displacement of the free end is U and velocity vanishes. Use the Finite Element Method on a regular spatial grid with $i \in \{0,1\}$ and the Discontinuous-Galerkin method with step size Δt to find the displacement and velocity of the free end at $t=\Delta t$. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

On a regular grid of points of spacing Δx, the generic equations for the string and bar problems according to the Finite Element Method are given by

$$\frac{k}{\Delta x}(a_{i-1}-2a_i+a_{i+1})+F_i+f'\Delta x=m'\frac{\Delta x}{6}(\ddot{a}_{i-1}+4\ddot{a}_i+\ddot{a}_{i+1}) \quad i \in \{1,2,\ldots,n-1\},$$

$$\frac{k}{\Delta x}(a_1-a_0)+F_0+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_0+\ddot{a}_1)=0 \quad \text{or} \quad a_0=\underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1}-a_n)+F_n+f'\frac{\Delta x}{2}-m'\frac{\Delta x}{6}(2\ddot{a}_n+\ddot{a}_{n-1})=0 \quad \text{or} \quad a_n=\underline{a}_n,$$

$$a_i-g_i=0 \quad \text{and} \quad \dot{a}_i-h_i=0.$$

In the bar problem show, $k=EA$, $m'=\rho A$ and the equations for points $i \in \{0,1\}$ become

$$u_0=0 \quad \text{and} \quad \frac{EA}{\Delta x}(u_0-u_1)-\rho A\frac{\Delta x}{6}(2\ddot{u}_1+\ddot{u}_0)=0 \quad t>0,$$

$$u_1=U \quad \text{and} \quad \dot{u}_1=0 \quad t=0.$$

The algebraic equation for point 0 can be used to eliminate u_0 from the differential equation for point 1. The outcome is the initial value problem

$$\frac{EA}{\Delta x}u_1+\rho A\frac{\Delta x}{3}\ddot{u}_1=0 \quad t>0, \quad u_1=U \quad t=0, \text{ and } \quad \dot{u}_1=0 \quad t=0.$$

After discretization with respect to the spatial coordinate, one may apply analytical solution method or discretize with respect to the temporal coordinate to get am algebraic equation system for typical time-step. Let us choose the latter approach and apply the Discontinuous-Galerkin method for a single equation

$$\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_i = \frac{2}{12+\alpha^4}\begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix}\begin{Bmatrix} a \\ \dot{a}\Delta t \end{Bmatrix}_{i-1}, \quad \begin{Bmatrix} a \\ \Delta t\dot{a} \end{Bmatrix}_0 = \begin{Bmatrix} g \\ \Delta t h \end{Bmatrix} \text{ where } \alpha = \sqrt{\frac{k}{m}}\Delta t\,.$$

With the present problem $a = u_1 = u$ so the iteration gives for the first step

$$\begin{Bmatrix} u \\ \dot{u}\Delta t \end{Bmatrix}_1 = \frac{2}{12+\alpha^4}\begin{bmatrix} 6-3\alpha^2 & 6-\alpha^2 \\ -6\alpha^2 & 6-3\alpha^2 \end{bmatrix}\begin{Bmatrix} U \\ 0 \end{Bmatrix} = \frac{2U}{12+\alpha^4}\begin{Bmatrix} 6-3\alpha^2 \\ -6\alpha^2 \end{Bmatrix} \text{ where } \alpha = \sqrt{3\frac{E}{\rho}}\frac{\Delta t}{\Delta x}.$$ ←