

## LECTURE ASSIGNMENT 1

Verify by direct calculation the weighted residual expression for the first derivative

$$\int_0^L N_i \frac{\partial a}{\partial x} dx = \frac{1}{2}(a_{i+1} - a_{i-1})$$

for a regular grid of spacing  $\Delta x$ . In a line segment of end points  $x_i$  and  $x_j$ , the non-zero linear shape functions and the interpolant  $a(x)$  are given by

$$\begin{Bmatrix} N_i \\ N_j \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ x_i & x_j \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} \quad \text{and} \quad a(x) = \begin{Bmatrix} a_i \\ a_j \end{Bmatrix}^T \begin{Bmatrix} N_i \\ N_j \end{Bmatrix}.$$

Place the origin of the  $x$ -coordinate system at point  $i$ .

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As the shape function  $N_i$  is non-zero only in the line segments having the point  $i$  in common and origin is placed at point  $i$ , it is enough to consider  $x \in [-\Delta x, \Delta x]$ . Using the expressions of the shape functions in line segments  $(i-1, i)$  and  $(i, i+1)$

$$x \in [-\Delta x, 0]: \begin{Bmatrix} N_{i-1} \\ N_i \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ -\Delta x & 0 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{Bmatrix} -x/\Delta x \\ x/\Delta x + 1 \end{Bmatrix} \text{ and } a(x) = \begin{Bmatrix} a_{i-1} \\ a_i \end{Bmatrix}^T \begin{Bmatrix} N_{i-1} \\ N_i \end{Bmatrix} \Rightarrow$$

$$N_i = 1 + \frac{x}{\Delta x}, \quad a(x) = -a_{i-1} \frac{x}{\Delta x} + a_i \frac{x + \Delta x}{\Delta x}, \text{ and } \frac{\partial a}{\partial x} = \frac{a_i - a_{i-1}}{\Delta x}$$

$$x \in [0, \Delta x]: \begin{Bmatrix} N_i \\ N_{i+1} \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \Delta x \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \end{Bmatrix} = \begin{Bmatrix} 1 - x/\Delta x \\ x/\Delta x \end{Bmatrix} \text{ and } a(x) = \begin{Bmatrix} a_i \\ a_{i+1} \end{Bmatrix}^T \begin{Bmatrix} N_i \\ N_{i+1} \end{Bmatrix} \Rightarrow$$

$$N_i = 1 - \frac{x}{\Delta x}, \quad a(x) = a_i \left(1 - \frac{x}{\Delta x}\right) + a_{i+1} \frac{x}{\Delta x}, \text{ and } \frac{\partial a}{\partial x} = \frac{a_{i+1} - a_i}{\Delta x}$$

Integral over the domain is the sum of the integrals over the line segments. Direct calculation gives first

$$\int_{-\Delta x}^0 N_i \frac{\partial a}{\partial x} dx = \frac{a_i - a_{i-1}}{2} \quad \text{and} \quad \int_0^{\Delta x} N_i \frac{\partial a}{\partial x} dx = \frac{a_{i+1} - a_i}{2}$$

and, after that, combining the integrals over the line segment having the point  $i$  in common ( $N_i$  vanishes elsewhere):

$$\int_0^L N_i \frac{\partial a}{\partial x} dx = \int_{-\Delta x}^0 N_i \frac{\partial a}{\partial x} dx + \int_0^{\Delta x} N_i \frac{\partial a}{\partial x} dx = \frac{1}{2}(a_{i+1} - a_{i-1}). \quad \leftarrow$$