Mathematics for Economists: Lecture 11

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Spring 2021

This lecture covers

1. What are dynamical systems?
2. Difference equations and motivating examples
3. Linear difference equations with constant coefficients

Dynamical systems

- Dynamical systems describe the evolution of variables over time
- For the state so the system today, x_t we can determine the state tomorrow x_{t+1}
$$x_{t+1} = f(x_t).$$
- The solution is a sequence $\{x_t\}$ satisfying this equation for all $t+1, t$.
- Hence the variable to be determined is the entire path of x.

Dynamical systems

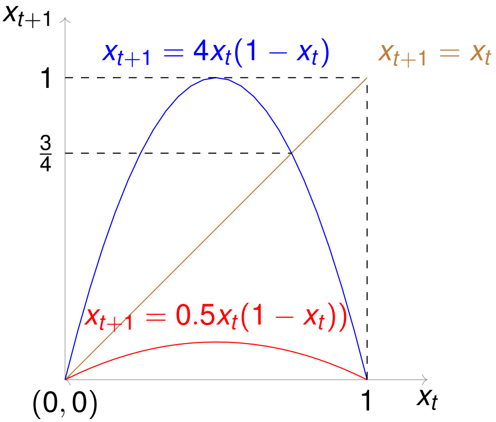
- What are the interesting questions with dynamical systems
- Is there a steady state, i.e. a value x^* such that $x^* = f(x^*)$?
- Do the solutions converge to this steady state?
- Are the solutions monotone?
- Can we have cycles?
- Because of time constraints, we cannot go very deep into this

Dynamical systems: logistic equation

- How to picture a dynamical system? We start with the case where $x_t \in \mathbb{R}$.
- Start with a concrete example:

$$x_{t+1} = rx_t(1 - x_t).$$

- This is a nice differentiable function whose values remain in $(0, 1)$ for all t as long as $r < 4$.
- To analyze a difference equation on the real line, the first step is to look at the graph of the system equation.

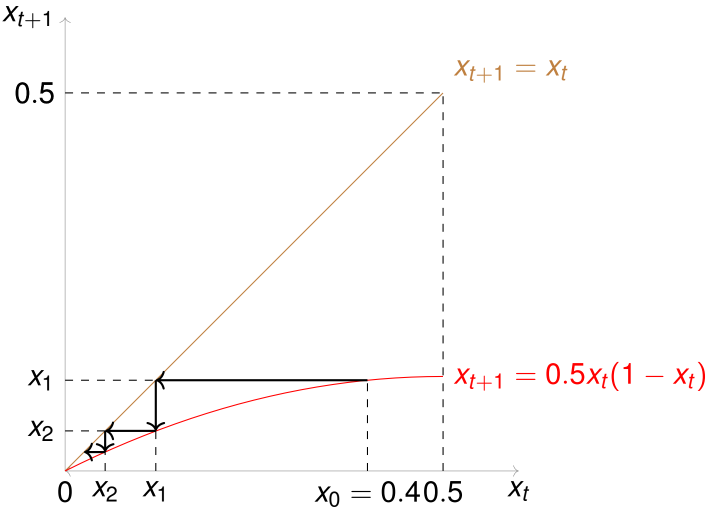

x_{t+1}
$x_{t+1} = 4x_t(1 - x_t)$
$x_{t+1} = x_t$
1
$\frac{3}{4}$
$x_{t+1} = 0.5x_t(1 - x_t))$
(0, 0)
1
x_t

Dynamical systems: logistic equation

- What is the significance of the intersections of $x_{t+1} = rx_t(1 - x_t)$ and the 45 -degree line $x_{t+1} = x_t$.
- The system stops at any such point, because if $x_{t+1} = x_t$, then also $x_{t+k} = x_t$ by repeated substitution into the system equation.
- These are called the steady states of the dynamical system.
- Notice that the system has a single steady state at $x = 0$ if $r < 1$ (can you show this?).
- For $4 > r > 1$, the system has another steady state at $x = \frac{r-1}{r}$. What happens to the values of x_t as t grows?

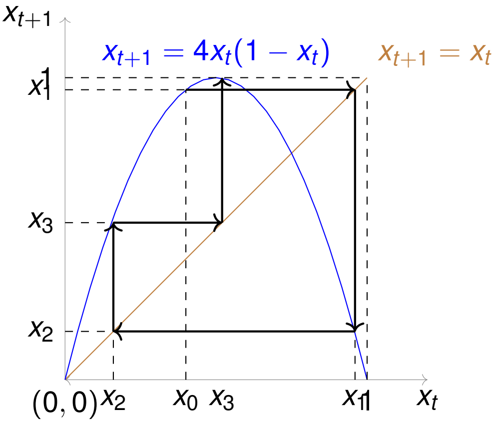
Dynamical systems: logistic equation

- Here is a nice graphical way of seeing what happens to the sequence.
- Lets graph the function in a coordinate system where x_{t+1} is on the vertical and x_t is on the horizontal axis.
- Draw the graph of $x_{t+1} = f(x_t)$ and pick a starting point $x_0 = 0.4$ for example on the horizontal axis.
- You can read $x_1 = f(x_0)$ on the graph.
- You need to picture x_1 on the horizontal axis to see where x_2 is located. But you can do this by reflection through the 45 -degree line. The you just continue the procedure.
- Lets look first at the case $x_{t+1} = \frac{1}{2}x_t(1 - x_t)$, i.e. lets take the red curve in the previous picture.


x_{t+1}
$x_{t+1} = x_t$
0.5
$x_{t+1} = 0.5x_t(1 - x_t)$
x_1
x_2
0
x_2
x_1
$x_0 = 0.4$
0.5
x_t

Dynamical systems: logistic equation

- As you can see, for any starting x_0, the system x_t converges quite quickly to 0.
- If we take the blue graph from the first picture, things look quite different.
- Let's follow the system again for a few rounds starting at $x_0 = 0.4$.



x_{t+1}
$x_{t+1} = 4x_t(1 - x_t)$
$x_{t+1} = x_t$
1
x_1
x_3
x_2
$(0, 0)$
x_2
x_0
x_3
x_1
1
x_t

Motivating examples: the Solow growth model

- In this simplest version of the model, labor is kept fixed at L over time and capital K_t changes over time as a result of savings.
- The aggregate production function is $y_t = F(K_t, L_t)$ and it is usually assumed to be an increasing concave, linearly homogenous (constant returns to scale) so that

$$F(K_t, L_t) = L_t F(\frac{K_t}{L_t}, 1) := L_t f(k_t) = L f(k_t),$$

where $k_t := \frac{K_t}{L_t}$.
- It is often assumed that $\lim_{k \to 0} f'(k) = \infty$ and $\lim_{k \to \infty} f'(k) = 0$.
- Let's assume here that $y_t = k_t^{\alpha}$ for some $0 < \alpha < 1$.

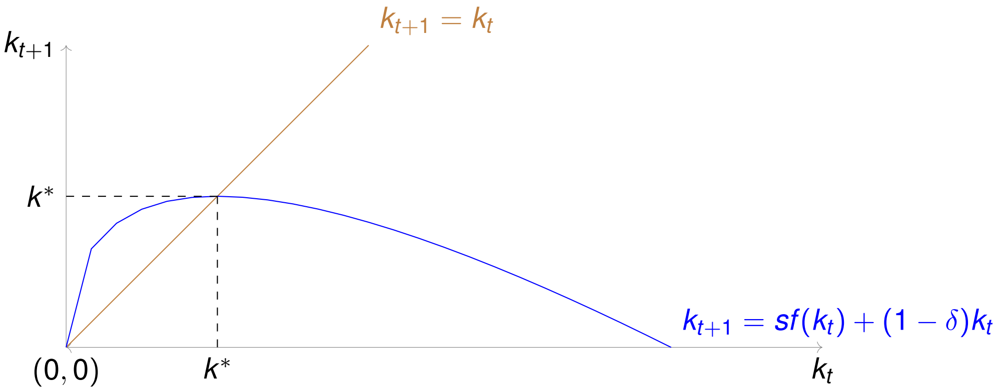
Motivating examples: the Solow growth model

- The output y_t is divided between savings and consumption.
- The assumption is that a constant fraction sy_t is saved.
- As you can recall from Principles of Economics II, savings equals investment and investment goes into next period's capital.
- Capital depreciates at rate δ per period.
- Taking all this together, we get

$$k_{t+1} = sk_t^{\alpha} + (1 - \delta)k_t.$$

Motivating examples: the Solow growth model

- Let's draw the graph of $sk_t^\alpha - (1-\delta)k_t$ in the (k_t, k_{t+1}) plane with the 45-degree line.
- Of special interest is the point k^* such that $s(k^*)^\alpha + (1-\delta)k^* = k^*$.
- If you start the system with $k_0 = k^*$, the system stays there forever since $k_1 = sk_0^\alpha + (1-\delta)k_0 = k_0$ and therefore also $k_n = k_0$ for all n.
- We call k^* the steady state or a rest point of the system.


$k_{t+1} = k_t$
k_{t+1}
k^*
$k_{t+1} = sf(k_t) + (1-\delta)k_t$
$(0,0)$
k^*
k_t

Motivating examples: the Solow growth model

- We can picture the movement of the system by positing an initial point k_0 on the horizontal axis.
- If $k_0 < k^*$, then $k_1 = f(k_0) > k_0$.
- You can locate the k_1 on the horizontal axis by reflecting on the 45-degree line.
- Repeating this process, you can show that from any initial point, k_t converges to k^* as $t \to \infty$.
- We say that k^* is a globally stable steady state.
- You may want to note that

$$k^* = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}.$$

Motivating examples: Ramsey-Cass-Koopmans model of growth

- Consider 2-dimensional system:

$$u'(c_t) = \beta(1 - \delta + f'(k_t))u'(c_{t+1}).$$

$$k_{t+1} = \frac{1}{1+n}\left(f(k_t) + (1-\delta)k_t - c_t\right),$$

- The first equation comes from consumer's optimal timing of their consumptions. Essentially it states that MRS equals price ration between any two periods of consumption.
- The second is a capital stock accumulation equation (in per capita term) as in the Solow model. This is the fundamental system for modern macroeconomics and the notes explain its solution in the linear case, where $u(c_t) = \ln c_t$, and $f(k_t) = Ak_t$.

Motivating examples: Fibonacci sequence

- Consider the sequence of numbers formed by the rule

$$x_{t+2} = x_{t+1} + x_t,$$

  and set $x_0 = 0, x_1 = 1$.
- What is the sequence of numbers generated by this rule?
- This sequence is called the Fibonacci sequence and it is one of the most famous sequences in all of mathematics
- Notice that we have now dependence on two past values, but we can deal with this by letting $y_{t+1} = x_t$ and considering the system

$$(x_{t+1}, y_{t+1}) = A(x_t, y_t)$$

  for a suitably chosen A.
- We'll see how to do this on Wednesday.

Motivating examples: SIR model of an epidemic

- Let the population be divided into three classes. Susceptible s_t, infected i_t, and recovered r_t.

$$
\begin{aligned}
s_{t+1} - s_t &= -\beta i_t s_t, \\
i_{t+1} - i_t &= \beta i_t s_t - \alpha i_t, \\
r_{t+1} - r_t &= \alpha i_t.
\end{aligned}
$$

- Here $\beta > 0$ is the infection rate in meetings and $\alpha > 0$ is the recovery rate.
- The much talked about R_0 is simply $\frac{\beta s_t}{\alpha}$.

Motivating examples: SIR model of an epidemic

- I guess I do not have to convince you of the importance of this model now.
- This is a non-linear model that does not have an easy closed form solution
- If $R_0 < 1$ at $t = 0$, then the number of infected is always decreasing and therefore there cannot be a proper outbreak. If $R_0 > 1$, a fraction of population will get infected and the fraction is increasing in R_0.
- You can see on Youtube a number of nice expositions on how to simulate an epidemic. For example a good one is on 3blue1brown at https://www.youtube.com/watch?v=gxAaO2rsdIs

Markov process

- A population consists of three income classes $i \in \{1, 2, 3\}$.
- If you are in class i, your children are in income class j with probability p_{ji}. Let P be the matrix with a typical element p_{ij}.
- Let $\boldsymbol{x}_0 = \mathbf{e^i}$ if you are in class i. Then the probability that you child is in class j is given by the column vector

$$\boldsymbol{x}_1 = P\boldsymbol{x}_0 = P\mathbf{e^i}.$$

- But then the probability that your grandchild is in class j is given by the column vector

$$\boldsymbol{x}_2 = P\boldsymbol{x}_1 = P^2\boldsymbol{x}_0,$$

and in general,

$$\boldsymbol{x}_{t+1} = P\boldsymbol{x}_t.$$

Markov process

- How do you interpret $\lim_{t\to\infty} P^t \boldsymbol{x}_0$?
- How do you describe a society with maximal (minimal) mobility?
- When does $\lim_{t\to\infty} P^t \boldsymbol{x}_0$ exist (aperiodicity of the process)? Is the limit independent of $\boldsymbol{x}_0$ (is the process mixing)?
- If $\boldsymbol{\pi} := (\pi_1, ..., \pi_n)$ solves
$$P\boldsymbol{\pi} = \boldsymbol{\pi},$$
does this mean that for all $\mathbf{e^i}$ the fraction of time spent in class j converges to π_j (is the process ergodic)?
- Is there a ranking on the matrices P that reflects the notion of persistence?

Linear difference equations in $\mathbb{R}$

- The simplest form of difference equations are linear difference equations with constant coefficients. These can be written as:

$$x_{t+1} = Ax_t + b_t,$$

  where b_t is a given sequence.
- If $b_t = 0$ for all t, we have a homogenous equation. We start with the simplest homogenous equations where $x_t \in \mathbb{R}$ and $A = a \in \mathbb{R}$.
- Solving the homogenous equation is very easy. If $x_{t+1} = ax_t$ for all t, then $x_{t+k} = a^k x_t$.
- Hence any sequence of the form $x_t = ca^t$ solves the difference equation.
- If we are given the initial value x_0, the solution is $x_t = x_0 a^t$.

Linear difference equations in $\mathbb{R}$

- In other words, the initial value pins down the coefficient c of the general solution.
- Consider next an inhomogenous equation,

$$x_{t+1} = ax_t + b,$$

  where $b_t = b$ for all t.
- Clearly the constant solution $x_t = \frac{b}{1-a}$ for all t solves the equation.
- I claim that also $x_t = ca^t + \frac{b}{1-a}$ solves the equation. But this follows immediately from the fact that $ca^{t+1} = aca^t$.
- This principle holds more generally. If you have a particular solution x_t^P to the inhomogenous equation and the general solution of the homogenous equation x_t^H, then the general solution to the problem is $x_t^P + x_t^H$.
- This is called the principle of superposition and it arises from the linearity of the equations in x_{t+1}, x_t. It is valid also for the case with $\boldsymbol{x}_t \in \mathbb{R}^n$.

Linear difference equations in $\mathbb{R}^n$

- If the following linear recursion holds for all t,
$$\boldsymbol{x}_{t+1} = \boldsymbol{A}\boldsymbol{x}_t,$$
then a solution to the model with $\boldsymbol{x}_0$ is easily obtained by repeated substitution:
$$\boldsymbol{x}_t = \boldsymbol{A}^t\boldsymbol{x}_0.$$
- Why do we call this a solution, but not the initial recursion?
- Why are we still not quite happy with this solution?
- Can you see what $\boldsymbol{A}^t$ looks like for large t?
- What if we can write
$$\boldsymbol{A} = \boldsymbol{V}^{-1}\boldsymbol{\Lambda}\boldsymbol{V},$$
for a diagonal matrix $\boldsymbol{\Lambda}$ and some matrix $\boldsymbol{V}$?
- Exercise: Prove by induction that in this case,
$$\boldsymbol{A}^k = \boldsymbol{V}^{-1}\boldsymbol{\Lambda}^k\boldsymbol{V}.$$
- The topic for the next lecture is how to find such $\boldsymbol{V}$ and $\boldsymbol{\Lambda}$?

Next Lecture

- General linear systems of difference equations with constant coefficients
- Eigenvalus, eigenvectors and matrix powers
- Qualitative properties of the solutions
- Recap of the course