Aalto University
Department of Mathematics and Systems Analysis
MS-A0211 Differential and Integral Calculus 2 (Radnell)

Exam 31.5.2021 from 9:00–13:00.

1. This exam consists of 6 problems, each of equal weight.
2. Any materials may be used during the exam.
3. Communication with anybody about the exam is strictly forbidden. All work must be your own
4. Answers must always be justified with calculations and explanations.
5. Homework grades from period III 2021 will be counted if they help, as per the grading rules for that course.

   Good luck!

1. When the 1st coin (black in the picture) rolls without sliding around the 2nd coin (blue in the picture) then the red dot marked on the edge of the smaller coin traces a plane curve (a type of epicycloid, dran in red) with parameterization

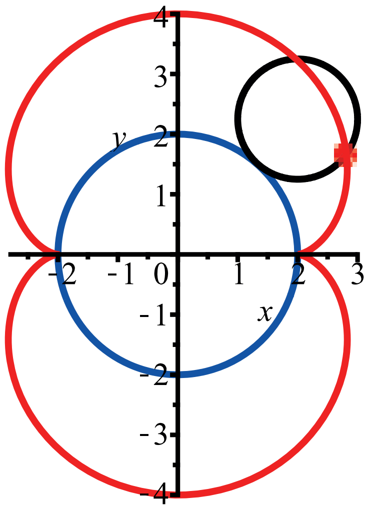
$$\begin{cases} x = 3\cos t - \cos(3t) \\ y = 3\sin t - \sin(3t), \end{cases}$$

for $t \in [0, 2\pi]$. Calculate the arc length of the curve for $t \in [0, \pi]$.

Hint: The identities $\sin^2 u + \cos^2 u = 1$ and

$$\cos t \cos(3t) + \sin t \sin(3t) = \cos(3t - t) = \cos(2t) = 1 - 2\sin^2 t.$$

may be useful.



2. Use linear approximation to estimate the value of

$$f(x, y) = \sqrt{2x^2 + e^{2y}}$$

at the point $(4.1, -0.1)$. Also, compare your answer to the value given by a calculator and discuss the accuracy of your result.

3. The point $P = (1, 1, 1)$ lies on the curve of intersection of the surfaces

$$F(x, y, z) = x^2 - xz + 2yz - 3y^2 + 1 = 0 \quad \text{and} \quad G(x, y, z) = x^3 + y^2 z - 2xz^2 = 0$$

The tangent line to the curve of intersection at the point P intersects the yz-plane (that is, the plane $x = 0$) at the point Q. Find the coordinates of Q.

Hint: First find the normal vectors to the surfaces at the point P.

4. Let $a > 0$, $b > 0$ and let $E_{a,b}$ be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The area of the ellipse is πab. Suppose the rectangle

$$\{(x, y) \in \mathbf{R}^2 \mid -1 \leq x \leq 1,\ -2 \leq y \leq 2)\}$$

lies inside the ellipse $E_{a,b}$. Find the values of a and b so that the ellipse has the smallest possible area. (That is, find the ellipse of smallest area that encloses the rectangle).

Note: You may assume (without justification) that the ellipse passes through the four vertices of the rectangle.

5. Consider the planar region $D = \{(x, y) \mid x \geq 0,\ 0 \leq y \leq e^{-x}\}$ of uniform density $\rho(x, y) = 1$. Find the *center of mass* of D. The area and coordinates of the center of mass are interpreted as improper integrals.

Note that in this example Fubini's theorem still applies to the improper double integrals and so you can use an improper iterated integrals to perform the calculations as usual.

6. Calculate the area of one loop the "propeller" given in polar coordinates by $0 \leq r \leq \sin(3\theta)$, for $0 \leq \theta \leq \pi/3$.