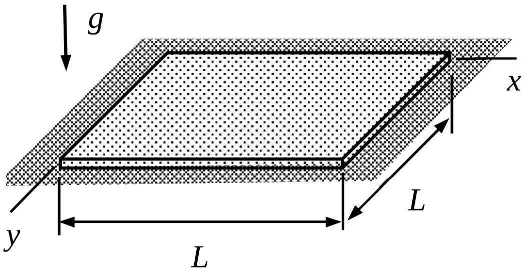
Name_________________________ Student number__________________

Home assignment 1

A rectangular membrane of side length L, density ρ, thickness t, and tightening S' (force per unit length) is loaded by its own weigh as shown. If the edges are fixed, find the transverse displacements at the grid points $(i,j) \in \{0,1,2,3,4\} \times \{0,1,2,3,4\}$ of a regular grid using the Finite Difference Method. Use symmetry to reduce the number of non-zero independent displacements to three.



Solution

The generic equations for the membrane model with fixed boundaries as given by the Finite Difference Method on a regular grid are

$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + f_i' = m_i' \ddot{w}_{(i,j)} \quad (i,j) \in I \,,$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I \,,$$

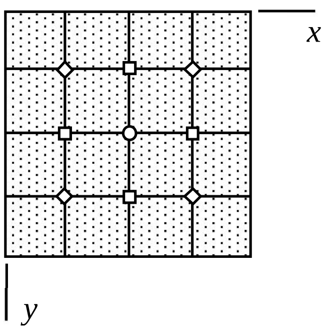
$$w_{(i,j)} - g_{(i,j)} = 0 \quad \text{and} \quad \dot{w}_{(i,j)} - h_{(i,j)} = 0 \qquad (i,j) \in I \,.$$

In the present problem, time derivatives vanish, initial conditions are not needed, and solution is reflection symmetric with respect to lines through the center point and aligned with the coordinate axes. Therefore, transverse displacements at the grid points satisfy

$$w_{(1,1)} = w_{(1,3)} = w_{(3,1)} = w_{(3,3)} = w_1 \,,$$

$$w_{(1,2)} = w_{(2,1)} = w_{(3,2)} = w_{(2,3)} = w_2 \,,$$

$$w_{(2,2)} = w_3 \,,$$



and the number of independent equilibrium equations is 3. Considering only the independent equations with $f' = \rho t g$ and $h = L/4$

$$16\frac{S'}{L^2}[w_{(0,1)}+w_{(1,0)}-4w_{(1,1)}+w_{(2,1)}+w_{(1,2)}]+\rho tg=16\frac{S'}{L^2}(-4w_1+2w_2)+\rho tg=0\,,$$

$$16\frac{S'}{L^2}[w_{(0,1)}+w_{(1,1)}-4w_{(1,2)}+w_{(2,2)}+w_{(1,3)}]+\rho tg=16\frac{S'}{L^2}(-4w_2+w_3+2w_1)+\rho tg=0\,,$$

$$16\frac{S'}{L^2}[w_{(1,2)}+w_{(2,1)}-4w_{(2,2)}+w_{(3,2)}+w_{(2,3)}]+\rho tg=16\frac{S'}{L^2}(-4w_3+4w_2)+\rho tg=0$$

or using the matrix notation

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & -4 & 4 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}-\frac{1}{16}\frac{\rho gtL^2}{S'}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}=0\,.$$

Then, using row operations to get an equivalent upper triangular matrix representation

$$\begin{bmatrix} 2 & -1 & 0 \\ -2 & 4 & -1 \\ 0 & -4 & 4 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}-\frac{1}{16}\frac{\rho gtL^2}{S'}\begin{Bmatrix} 1/2 \\ 1 \\ 1 \end{Bmatrix}=0 \;\Leftrightarrow\; \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & -4 & 4 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}-\frac{1}{16}\frac{\rho gtL^2}{S'}\begin{Bmatrix} 1/2 \\ 3/2 \\ 1 \end{Bmatrix}=0 \;\Leftrightarrow$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 12 & -4 \\ 0 & -12 & 12 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}-\frac{1}{16}\frac{\rho gtL^2}{S'}\begin{Bmatrix} 1/2 \\ 6 \\ 3 \end{Bmatrix}=0 \;\Leftrightarrow\; \begin{bmatrix} 2 & -1 & 0 \\ 0 & 12 & -4 \\ 0 & 0 & 8 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}-\frac{1}{16}\frac{\rho gtL^2}{S'}\begin{Bmatrix} 1/2 \\ 6 \\ 9 \end{Bmatrix}=0\,.$$

Then, using the equations starting from the last one

$$w_3=\frac{9}{128}\frac{\rho gtL^2}{S'}\,,\quad w_2=\frac{7}{128}\frac{\rho gtL^2}{S'}\,,\ \text{and}\ w_1=\frac{11}{256}\frac{\rho gtL^2}{S'}\,.$$