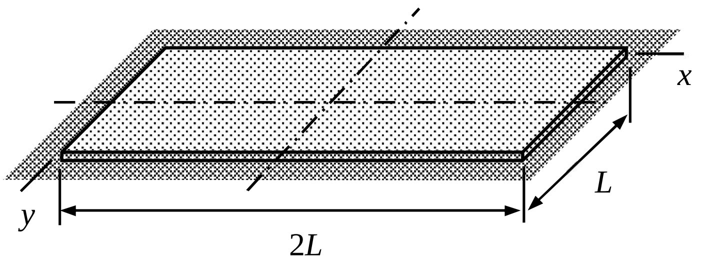
Name________________________ Student number________________

Home assignment 2

Consider vibration of a rectangular membrane of side lengths $2L$ and L, density ρ, thickness t, and tightening S' (force per unit length). If the edges are fixed, find the angular velocity of the free vibrations using the Finite Difference Method on a regular grid of points $(i,j) \in \{0,1,2,3,4\} \times \{0,1,2\}$. Consider the modes, that are reflection symmetric with respect to the lines through the center point (figure).



Solution

The generic equations for the membrane model with fixed boundaries as given by the Finite Difference Method on a regular grid are

$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + f' = m'\ddot{w}_{(i,j)} \quad (i,j) \in I,$$

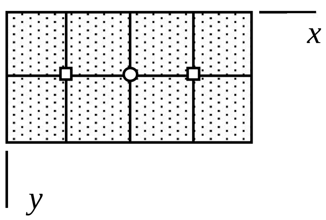
$$w_{(i,j)} = 0 \quad (i,j) \in \partial I,$$

$$w_{(i,j)} - g_{(i,j)} = 0 \quad \text{and} \quad \dot{w}_{(i,j)} - h_{(i,j)} = 0 \quad (i,j) \in I.$$

In modal analysis, initial conditions are not needed. As the mode is assumed to be reflection symmetric with respect to lines through the center point and aligned with the coordinate axes, transverse displacements at the grid points satisfy

$$w_{(1,1)} = w_{(3,1)} = w_1,$$

$$w_{(2,1)} = w_2,$$



the remaining displacements at the boundary points being zeros. Considering only the independent equations with $f' = 0$, $m' = \rho t$, and $h = L/2$

$$4\frac{S'}{L^2}[w_{(0,1)} + w_{(1,0)} - 4w_{(1,1)} + w_{(2,1)} + w_{(1,2)}] = 4\frac{S'}{L^2}(-4w_1 + w_2) = \rho t \ddot{w}_1,$$

$$4\frac{S'}{L^2}[w_{(1,1)}+w_{(2,0)}-4w_{(2,1)}+w_{(3,1)}+w_{(2,2)}]=4\frac{S'}{L^2}(-4w_2+2w_1)=\rho t\ddot{w}_2$$

or using the matrix notation

$$\begin{bmatrix} 4 & -1 \\ -2 & 4 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}+\frac{1}{4}\frac{\rho t L^2}{S'}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix}=0 .$$

Solution to the angular velocities and the corresponding modes follow with the trial solution $\mathbf{a}=\mathbf{A}e^{\mathrm{i}\omega t}$:

$$(\begin{bmatrix} 4 & -1 \\ -2 & 4 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \text{ where } \lambda=\omega^2\frac{1}{4}\frac{\rho t L^2}{S'} \quad\Leftrightarrow\quad \omega=\frac{2}{L}\sqrt{\lambda\frac{S'}{\rho t}} .$$

A homogeneous linear equation system can yield a non-zero solution to the mode only if the matrix is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

$$\det\begin{bmatrix} 4-\lambda & -1 \\ -2 & 4-\lambda \end{bmatrix}=(4-\lambda)^2-2=0 \text{ so } \lambda_1=4-\sqrt{2} \text{ or } \lambda_2=4+\sqrt{2} .$$

Knowing the possible values for a non-zero solution, the modes follow from the algebraic equation when the values of parameter λ are substituted there (one at a time):

$$\lambda_1=4-\sqrt{2}:\ \omega_1=\frac{2}{L}\sqrt{(4-\sqrt{2})\frac{S'}{\rho t}} \text{ and } \begin{bmatrix} \sqrt{2} & -1 \\ -2 & \sqrt{2} \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \text{ so}$$

$$(\omega_1,\mathbf{A}_1)=(\frac{2}{L}\sqrt{(4-\sqrt{2})\frac{S'}{\rho t}},\begin{Bmatrix} 1 \\ \sqrt{2} \end{Bmatrix}) . \leftarrow$$

$$\lambda_2=4+\sqrt{2}:\ \omega_2=\frac{2}{L}\sqrt{(4+\sqrt{2})\frac{S'}{\rho t}} \text{ and } \begin{bmatrix} -\sqrt{2} & -1 \\ -2 & -\sqrt{2} \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \text{ so}$$

$$(\omega_2,\mathbf{A}_2)=(\frac{2}{L}\sqrt{(4+\sqrt{2})\frac{S'}{\rho t}},\begin{Bmatrix} 1 \\ -\sqrt{2} \end{Bmatrix}) . \leftarrow$$