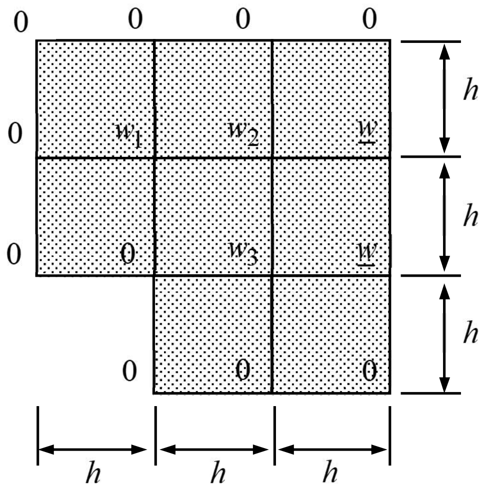
Name________________________ Student number________________

Home assignment 3

Consider the membrane of polygonal shape of the figure for which density ρ, thickness t, tightening S' (force per unit length) are constants. Using the given values at the grid points on the boundary, determine the unknown transverse displacement values w_1, w_2 and w_3 using the Finite Difference Method.



Solution

The generic equations for the membrane model with fixed boundaries as given by the Finite Difference Method on a regular grid are

$$\frac{S'}{h^2}[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + f_i' = m_i' \ddot{w}_{(i,j)} \quad (i,j) \in I \,,$$

$$w_{(i,j)} = 0 \quad (i,j) \in \partial I \,,$$

$$w_{(i,j)} - g_{(i,j)} = 0 \quad \text{and} \quad \dot{w}_{(i,j)} - h_{(i,j)} = 0 \qquad (i,j) \in I \,.$$

In the present problem, time derivatives vanish and initial conditions are not needed. The equilibrium equations for the interior points are

$$\frac{S'}{h^2}(-4w_1 + w_2) = 0 \,, \quad \frac{S'}{h^2}(-4w_2 + w_1 + w_3 + \underline{w}) = 0 \,, \text{ and } \frac{S'}{h^2}(-4w_3 + w_2 + \underline{w}) = 0 \,.$$

Using the matrix notation

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \underline{w} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = 0 \,.$$

Then, using row operations to get an equivalent upper triangular matrix representation

$$\begin{bmatrix} 4 & -1 & 0 \\ -4 & 16 & -4 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \underline{w} \begin{Bmatrix} 0 \\ 4 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 4 & -1 & 0 \\ 0 & 15 & -4 \\ 0 & -1 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \underline{w} \begin{Bmatrix} 0 \\ 4 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{bmatrix} 4 & -1 & 0 \\ 0 & 15 & -4 \\ 0 & -15 & 60 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \underline{w} \begin{Bmatrix} 0 \\ 4 \\ 15 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 4 & -1 & 0 \\ 0 & 15 & -4 \\ 0 & 0 & 56 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \underline{w} \begin{Bmatrix} 0 \\ 4 \\ 19 \end{Bmatrix} = 0 .$$

After that, equations can be solved one-by-one for the displacement starting from the last one

$$w_3 = \frac{19}{56}\underline{w}, \quad w_2 = \frac{5}{14}\underline{w}, \text{ and } w_1 = \frac{5}{56}\underline{w}.$$