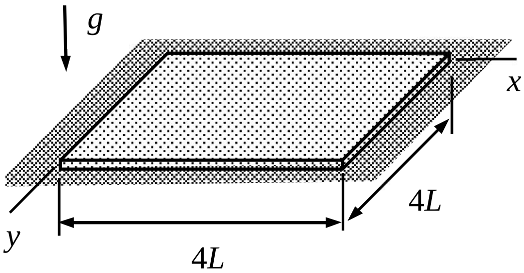
Name_________________________ Student number_________________

Home assignment 1

A rectangular membrane of side length $4L$, density ρ, thickness t, and tightening S' (force per unit length) is loaded by its own weigh as shown. If the edges are fixed, find the transverse displacements at the grid points $(i, j) \in \{0,1,2,3,4\} \times \{0,1,2,3,4\}$ of constant spacing. Use the Finite Element Method with a piecewise linear approximation on regular triangle elements. Use symmetry to reduce the number of non-zero independent displacements to three.



Solution

In stationary problem, the generic equations for the membrane model with fixed boundaries as given by the Finite Element Method on a regular grid are

$$S'[w_{(i-1,j)} + w_{(i,j-1)} - 4w_{(i,j)} + w_{(i+1,j)} + w_{(i,j+1)}] + h^2 f' = 0 \quad (i, j) \in I \,,$$

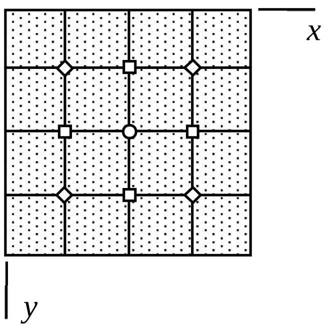
$$w_{(i,j)} = 0 \quad (i, j) \in \partial I \,,$$

In the present problem, time derivatives vanish, initial conditions are not needed, and solution is reflection symmetric with respect to lines through the center point and aligned with the coordinate axes. Therefore, transverse displacements at the grid points satisfy

$$w_{(1,1)} = w_{(1,3)} = w_{(3,1)} = w_{(3,3)} = w_1 \,,$$

$$w_{(1,2)} = w_{(2,1)} = w_{(3,2)} = w_{(2,3)} = w_2 \,,$$

$$w_{(2,2)} = w_3 \,,$$



and the number of independent equilibrium equations is 3. According to the principle of virtual work, the equation for a constrained displacement is the sum of equations for the constrained points. In a stationary problem, it is enough to consider just one of them as the equations coincide. Considering only the equations for points $(1,1)$, $(1,2)$, and $(2,2)$ with $f' = \rho t g$ and $h = L$

$$S'[w_{(0,1)} + w_{(1,0)} - 4w_{(1,1)} + w_{(2,1)} + w_{(1,2)}] + L^2\rho tg = S'(-4w_1 + 2w_2) + L^2\rho tg = 0\,,$$

$$S'[w_{(0,1)} + w_{(1,1)} - 4w_{(1,2)} + w_{(2,2)} + w_{(1,3)}] + L^2\rho tg = S'(-4w_2 + w_3 + 2w_1) + L^2\rho tg = 0\,,$$

$$S'[w_{(1,2)} + w_{(2,1)} - 4w_{(2,2)} + w_{(3,2)} + w_{(2,3)}] + L^2\rho tg = S'(-4w_3 + 4w_2) + L^2\rho tg = 0$$

or using the matrix notation

$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -1 \\ 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \frac{\rho gtL^2}{S'} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 0\,.$$

Then, using row operations to get an equivalent upper triangular matrix representation

$$\begin{bmatrix} 2 & -1 & 0 \\ -2 & 4 & -1 \\ 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \frac{\rho gtL^2}{S'} \begin{Bmatrix} 1/2 \\ 1 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & -4 & 4 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \frac{\rho gtL^2}{S'} \begin{Bmatrix} 1/2 \\ 3/2 \\ 1 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 12 & -4 \\ 0 & -12 & 12 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \frac{\rho gtL^2}{S'} \begin{Bmatrix} 1/2 \\ 6 \\ 3 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 12 & -4 \\ 0 & 0 & 8 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - \frac{\rho gtL^2}{S'} \begin{Bmatrix} 1/2 \\ 6 \\ 9 \end{Bmatrix} = 0\,.$$

Then, using the equations starting from the last one

$$w_3 = \frac{9}{8}\frac{\rho gtL^2}{S'}\,, \quad w_2 = \frac{7}{8}\frac{\rho gtL^2}{S'}\,, \text{ and } w_1 = \frac{11}{16}\frac{\rho gtL^2}{S'}\,.$$