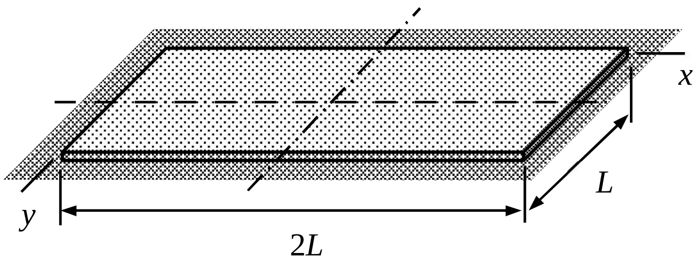
Name________________________ Student number________________

Home assignment 2

Consider free vibration of a rectangular membrane of side lengths $2L$ and L, density ρ, thickness t, and tightening S'. If the edges are fixed, find the angular velocities of the free vibrations using a regular grid $(i,j)\in\{0,1,2,3,4\}\times\{0,1,2\}$ of constant spacing and the Finite Element Method with a piecewise linear approximation on regular triangle elements. Consider the modes, that are reflection symmetric with respect to the lines through the center point (figure).



Solution

Assuming regular grid of points and regular triangle division, the generic equations for the membrane model with fixed boundaries as given by the Finite Element Method are

$$S'[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+h^2 f'=$$

$$m'h^2\frac{1}{12}[\ddot{w}_{(i-1,j-1)}+\ddot{w}_{(i-1,j)}+\ddot{w}_{(i,j-1)}+6\ddot{w}_{(i,j)}+\ddot{w}_{(i+1,j)}+\ddot{w}_{(i,j+1)}+\ddot{w}_{(i+1,j+1)}] \quad (i,j)\in I \quad t>0$$

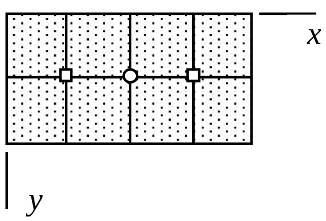
$$w_{(i,j)}=0 \quad (i,j)\in\partial I \quad t>0\,,$$

$$w_{(i,j)}-g_{(i,j)}=0 \quad \text{and} \quad \dot{w}_{(i,j)}-h_{(i,j)}=0 \quad (i,j)\in I \quad t=0\,.$$

In modal analysis, initial conditions are not needed. As the mode is assumed to be reflection symmetric with respect to lines through the center point and aligned with the coordinate axes, transverse displacements at the grid points satisfy

$$w_{(1,1)}=w_{(3,1)}=w_1\,,$$

$$w_{(2,1)}=w_2\,,$$



the remaining displacements at the boundary points being zeros. According to the principle of virtual work, the equation for a constrained displacement is the sum of equations for the constrained points. In the present problem, it is enough to consider just one of them as the equations of points $(1,1)$ and

(3,1) coincide. Considering only the equations of points (1,1) and (2,1) with $f'=0$, $m'=\rho t$, and $h=L/2$

$$S'(-4w_1+w_2)=\rho t L^2\frac{1}{48}(6\ddot{w}_1+\ddot{w}_2),$$

$$S'(-4w_2+2w_1)=\rho t L^2\frac{1}{48}(6\ddot{w}_2+2\ddot{w}_1)$$

or using the matrix notation

$$\begin{bmatrix} 4 & -1 \\ -2 & 4 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix}+\frac{1}{48}\frac{\rho t L^2}{S'}\begin{bmatrix} 6 & 1 \\ 2 & 6 \end{bmatrix}\begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix}=0.$$

Solution to the angular velocities and the corresponding modes follow with the trial solution $\mathbf{a}=\mathbf{A}e^{\mathrm{i}\omega t}$:

$$(\begin{bmatrix} 4 & -1 \\ -2 & 4 \end{bmatrix}-\lambda\begin{bmatrix} 6 & 1 \\ 2 & 6 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}=0 \quad \text{where} \quad \lambda=\omega^2\frac{1}{48}\frac{\rho t L^2}{S'} \quad \Leftrightarrow \quad \omega=\frac{4}{L}\sqrt{3\lambda\frac{S'}{\rho t}}.$$

A homogeneous linear equation system can yield a non-zero solution to the mode only if the matrix is singular, i.e., its determinant vanishes. The condition can be used to find the possible values of λ

$$\det\begin{bmatrix} 4-6\lambda & -1-\lambda \\ -2-2\lambda & 4-6\lambda \end{bmatrix}=(4-6\lambda)^2-2(1+\lambda)^2=14-52\lambda+34\lambda^2=0 \quad \Rightarrow$$

$$\lambda_1=\frac{1}{17}(13-5\sqrt{2}) \quad \text{or} \quad \lambda_2=\frac{1}{17}(13+5\sqrt{2}).$$

Knowing the possible values of parameter λ, the angular velocities follow from the relationship between λ and ω:

$$\lambda_1=\frac{1}{17}(13-5\sqrt{2}):\quad \omega_1=\frac{4}{L}\sqrt{\frac{3}{17}(13-5\sqrt{2})\frac{S'}{\rho t}}, \quad \Leftarrow$$

$$\lambda_2=\frac{1}{17}(13+5\sqrt{2}):\quad \omega_2=\frac{4}{L}\sqrt{\frac{3}{17}(13+5\sqrt{2})\frac{S'}{\rho t}}. \quad \Leftarrow$$