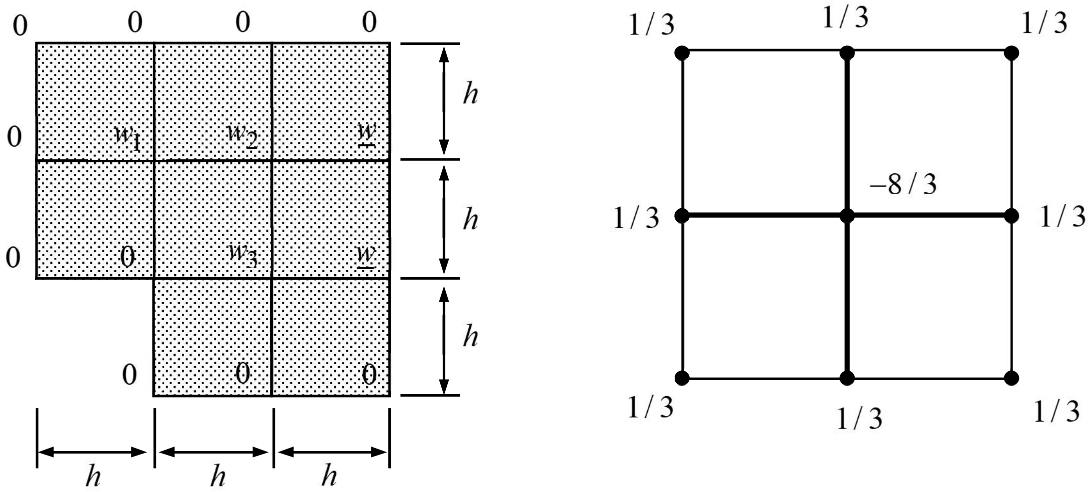
Name________________________ Student number________________

Home assignment 3

Consider the membrane of polygonal shape of the figure for which density ρ, thickness t, tightening S' (force per unit length) are constants. Using the given displacement values at the grid points on the boundary, determine the unknown transverse displacement values w_1, w_2 and w_3 using the given Laplacian stencil for the interior points given by the Finite Element Method and bi-linear approximation on rectangle elements.



Solution

The equations for the interior points can be obtained by centering the stencil at the grip one at a time and using the multipliers as the weighting of the function values. The outcomes for the interior points 1, 2, and 3 are (notice that the values at the boundary points are known)

$$\frac{S'}{3}(-8w_1 + w_2 + w_3) = 0\,, \quad \frac{S'}{3}(-8w_2 + w_1 + w_3 + 2\underline{w}) = 0\,, \text{ and } \frac{S'}{3}(-8w_3 + w_1 + w_2 + 2\underline{w}) = 0$$

Using the matrix notation

$$\begin{bmatrix} 8 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & 8 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - 2\underline{w} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} = 0\,.$$

Then, using row operations to get an equivalent upper triangular matrix representation

$$\begin{bmatrix} 8 & -1 & -1 \\ -8 & 64 & -8 \\ -8 & -8 & 64 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - 2\underline{w} \begin{Bmatrix} 0 \\ 8 \\ 8 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 8 & -1 & -1 \\ 0 & 63 & -9 \\ 0 & -9 & 63 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - 2\underline{w} \begin{Bmatrix} 0 \\ 8 \\ 8 \end{Bmatrix} = 0 \quad \Leftrightarrow$$

$$\begin{bmatrix} 8 & -1 & -1 \\ 0 & 63 & -9 \\ 0 & -63 & 441 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - 2\underline{w} \begin{Bmatrix} 0 \\ 8 \\ 56 \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{bmatrix} 8 & -1 & -1 \\ 0 & 63 & -9 \\ 0 & 0 & 432 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} - 2\underline{w} \begin{Bmatrix} 0 \\ 8 \\ 64 \end{Bmatrix} = 0 .$$

After that, equations can be solved one-by-one for the displacement starting from the last one

$$w_3 = \frac{8}{27}\underline{w}, \quad w_2 = \frac{8}{27}\underline{w}, \text{ and } w_1 = \frac{2}{27}\underline{w}.$$