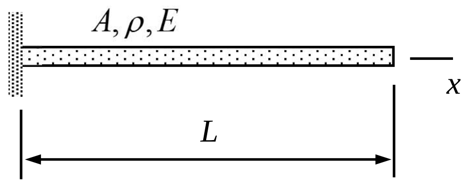
COE-C3005 Finite Element and Finite difference methods, Remote exam 04.06.2021

Problem 1

Determine the angular velocities of the free vibrations of the bar shown by using the Finite Element Method on a regular grid with $i \in \{0,1,2\}$. Cross-sectional area A, density ρ of the material, and Young's modulus E of the material are constants.



Solution

The difference equations to the bar and string problems according to the Finite Element Method are given by (omitting the initial conditions as they are needed in modal analysis)

$$\frac{k}{\Delta x}(a_{i-1} - 2a_i + a_{i+1}) + F_i + f'\Delta x = m'\frac{\Delta x}{6}(\ddot{a}_{i-1} + 4\ddot{a}_i + \ddot{a}_{i+1}) \qquad i \in \{1,2,\ldots,n-1\}$$

$$\frac{k}{\Delta x}(a_1 - a_0) + F_0 + f'\frac{\Delta x}{2} - m'\frac{\Delta x}{6}(2\ddot{a}_0 + \ddot{a}_1) = 0 \quad \text{or} \quad a_0 = \underline{a}_0,$$

$$\frac{k}{\Delta x}(a_{n-1} - a_n) + F_n + f'\frac{\Delta x}{2} - m'\frac{\Delta x}{6}(2\ddot{a}_n + \ddot{a}_{n-1}) = 0 \quad \text{or} \quad a_n = \underline{a}_n.$$

Where in this case of a bar problem $k = EA$, $m' = \rho A, \Delta x = L/2$, $a = u$ and external forces vanish. Equations for the three grid points are

$$u_0 = 0,$$

$$2\frac{AE}{L}(u_0 - 2u_1 + u_2) - \rho A\frac{L}{12}(\ddot{u}_0 + 4\ddot{u}_1 + \ddot{u}_2) = 0,$$

$$2\frac{EA}{L}(u_1 - u_2) - \rho A\frac{L}{12}(2\ddot{u}_2 + \ddot{u}_1) = 0.$$

After using the first algebraic equation to eliminate u_0 from the ordinary differential equations, the matrix representation, required by the modal analysis, takes the form

$$2\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} + \rho A\frac{L}{12}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = 0 .$$

Modal analysis uses the trial solution $\mathbf{u} = \mathbf{A}e^{\mathrm{i}\omega t}$ in which $\mathbf{A}$ represent mode and ω the corresponding angular velocity. Substitution into the set of differential equations gives a set of algebraic equations for the possible combinations $(\omega, \mathbf{A})$:

$$(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix})\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = 0 \quad \text{where} \quad \lambda = \frac{1}{24}\frac{\rho L^2}{E}\omega^2 \quad \Leftrightarrow \quad \omega = \frac{2}{L}\sqrt{6\lambda\frac{E}{\rho}} .$$

First, the possible λ values:

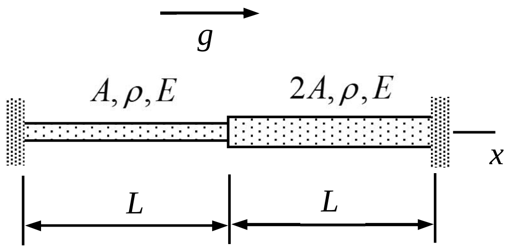
$$\det(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}) = 2(1-2\lambda)^2 - (1+\lambda)^2 = 0 \quad \Rightarrow \quad \lambda = \frac{1}{7}(5 \pm 3\sqrt{2}) .$$

Therefore, the angular velocities of the free vibrations

$$\omega = \frac{2}{L}\sqrt{\frac{6}{7}(5 \pm 3\sqrt{2})\frac{E}{\rho}} . \quad \leftarrow$$

Problem 2

The stationary boundary value problem for the bar shown (jump in the cross-sectional area at $x = L$) consists of the equilibrium equations for the regular interior points, jump condition at $x = L$, and displacement boundary conditions for the end points. Write the equation system $-\mathbf{K}\mathbf{u} + \mathbf{F} = 0$ according to the Finite Difference Method on a regular grid with $i \in \{0,1,2,3,4\}$ and solve for the displacements. Use the proper forward/backward difference approximation to the first derivative in the jump condition. Young's modulus E and density ρ of the material are constants.



Solution

The zero displacement boundary conditions for points 0 and 4 are obvious, The difference equations for the regular interior points 2 and 3, follow when the second order derivative in the stationary differential equation is replaced by central difference approximation and the distributed load due to gravity is evaluated at those points. In the jump condition for the cemterpoint, the first derivatives are replaced by backward and forward approximations:

$$u_0 = 0,$$

$$\frac{EA}{\Delta x^2}(u_0 - 2u_1 + u_2) + \rho g A = 0,$$

$$\frac{E2A}{\Delta x}(u_3 - u_2) - \frac{EA}{\Delta x}(u_2 - u_1) = 0,$$

$$\frac{E2A}{\Delta x^2}(u_2 - 2u_3 + u_4) + \rho g 2A = 0,$$

$$u_4 = 0.$$

Next, using the boundary conditions to eliminate u_0 and u_4, the remaining equations can be written in the matrix form

$$-\frac{EA}{L^2}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 4 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \rho A g \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix} = 0.$$

Solution can be obtained by (Gauss) elimination (or using Mathematica). First, row operations to get an equivalent upper diagonal form:

$$\begin{bmatrix} 2 & -1 & 0 \\ -2 & 6 & -4 \\ 0 & -2 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 1 \\ 0 \\ 2 \end{Bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 5 & -4 \\ 0 & -2 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix},$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 5 & -4 \\ 0 & -2 & 4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & -1 & 0 \\ 0 & 5 & -4 \\ 0 & 3 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 1 \\ 1 \\ 3 \end{Bmatrix}.$$

Then, considering the equations one-by-one starting from the last one

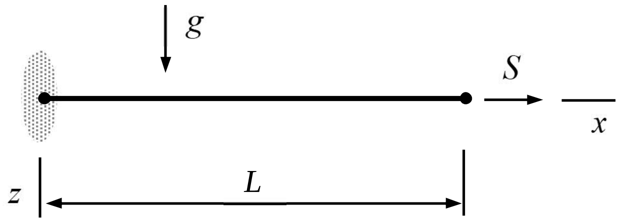
$$u_2 = \frac{\rho g L^2}{E} \quad \Rightarrow \quad u_3 = -\frac{1}{4}(\frac{\rho g L^2}{E} - 5u_2) = \frac{\rho g L^2}{E} \quad \Rightarrow \quad u_1 = \frac{1}{2}(\frac{\rho g L^2}{E} + u_2) = \frac{\rho g L^2}{E}.$$

Altogether

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\rho g L^2}{E} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}. \quad \leftarrow$$

Problem 3

A string of length L, density ρ, cross-sectional area A, and tightening S is loaded by its own weight as shown. If the left end is fixed, the right end is free, and the initial geometry without loading is straight, find the stationary transverse displacement according to the Finite Element Method with regular grid $i \in \{0,1,\ldots,n\}$. What is the limit solution when $n \to \infty$?



Solution

In the stationary case all time derivatives vanish and initial conditions are not needed. The generic equation set for the string and bar models is given the Finite Element Method on a regular grid simplifies to

$$\frac{S}{\Delta x}(w_{i-1} - 2w_i + w_{i+1}) + \rho A g \Delta x = 0 \quad i \in \{1,2,\ldots,n-1\},$$

$$w_0 = 0 \quad \text{and} \quad \frac{S}{\Delta x}(w_{n-1} - w_n) + \frac{\Delta x}{2}\rho g A = 0,$$

where $\Delta x = L / n$. Let us find first the generic solution to the difference equation for the interior points using the same approach as with differential equations. The generic solution is composed of the generic solution to the homogeneous equation of the form $w_h = Ar^i$ to the homogeneous equation (loading omitted) and particular solution $w_p = Ci^2$. Substituting the trial solution w_h

$$Ar^{i-1} - 2Ar^i + Ar^{i+1} = (1 - 2r + r^2)Ar^{i-1} = (1-r)^2 Ar^{i-1} = 0.$$

Due to the double root $r = 1$, one obtains $w_h = A + Bi$. The particular solution follows with the solution trial $w_p = Ci^2$. Substitution into the (full) equilibrium equation gives

$$\frac{S}{\Delta x}[C(i-1)^2 - 2Ci^2 + C(i+1)^2] + \rho A g \Delta x = 0 \quad \Rightarrow \quad C = -\frac{\rho A g}{2S}\Delta x^2.$$

The generic solution to the difference equation for the interior points

$$w_i = w_h + w_p = A + Bi - \frac{\rho A g \Delta x^2}{2S} i^2$$

contains parameters A and B to be determined from the equations for the boundary points.

$w_0 = A = 0$ and $B = \frac{\rho A g \Delta x^2}{S} n$.

Therefore, finally

$$w_i = \frac{\rho A g \Delta x^2}{2S} i(2n - i) .$$

The limit solution can be obtained by writing the solution first into the form with notation $x_i = \Delta x i$

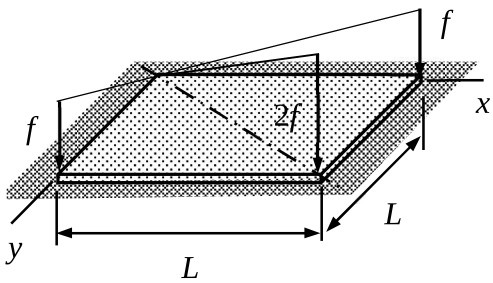
$$w_i = \frac{\rho A g}{2S} x_i (2L - x_i)$$

and considering $n \to \infty$ meaning that $\Delta x = L / n \to 0$. Then labelling the points with indices is not possible and material coordinate x needs to be used for particle identification:

$$w(x) = \frac{\rho A g}{2S} x(2L - x) .$$

Problem 4

A rectangular membrane of side length L and tightening S' is loaded by a non-constant distributed force $f'(x,y)=(x+y)f/L$ in which f is constant. If the edges are fixed, find the transverse displacement using the Finite Difference Method on a regular grid $(i,j)\in\{0,1,2,3\}\times\{0,1,2,3\}$ and reflection symmetry with respect to the line shown in figure.



Solution

The generic equations for the membrane model with fixed boundaries, as given by the Finite Difference Method on a regular grid, are

$$\frac{S'}{h^2}[w_{(i-1,j)}+w_{(i,j-1)}-4w_{(i,j)}+w_{(i+1,j)}+w_{(i,j+1)}]+f'=m'\ddot{w}_{(i,j)}\quad (i,j)\in I\,,$$

$$w_{(i,j)}=0\quad (i,j)\in\partial I\,,$$

$$w_{(i,j)}-g_{(i,j)}=0\quad\text{and}\quad \dot{w}_{(i,j)}-h_{(i,j)}=0\qquad (i,j)\in I\,.$$

In the present problem, time derivatives vanish, initial conditions are not needed, and solution is reflection symmetric with respect to line shown in the figure. Therefore, transverse displacements at the interior grid points satisfy

$$w_{(1,1)}=w_1\,,\ w_{(1,2)}=w_{(2,1)}=w_2\,,\ \text{and}\ \ w_{(2,2)}=w_3\,.$$

Let us write the equilibrium equations for the interior points one-by-one with the displacement constraints. In the Finite Difference Method, external distributed force is evaluated at the grid points. Here $h=L/3$:

$$9\frac{S'}{L^2}[w_{(0,1)}+w_{(1,0)}-4w_{(1,1)}+w_{(2,1)}+w_{(1,2)}]+\frac{2}{3}f=0\quad\Rightarrow\quad 9\frac{S'}{L^2}(-4w_1+2w_2)+\frac{2}{3}f=0\,,$$

$$9\frac{S'}{L^2}[w_{(0,2)}+w_{(1,1)}-4w_{(1,2)}+w_{(2,2)}+w_{(1,3)}]+f=0\quad\Rightarrow\quad 9\frac{S'}{L^2}(w_1-4w_2+w_3)+f=0\,,$$

$$9\frac{S'}{L^2}[w_{(1,1)}+w_{(2,0)}-4w_{(2,1)}+w_{(3,1)}+w_{(2,2)}]+f=0\quad\Rightarrow\quad 9\frac{S'}{L^2}(w_1-4w_2+w_3)+f=0\,,$$

$$9\frac{S'}{L^2}[w_{(1,2)}+w_{(2,1)}-4w_{(2,2)}+w_{(3,2)}+w_{(2,3)}]+\frac{4}{3}f=0 \quad \Rightarrow \quad 9\frac{S'}{L^2}(2w_2-4w_3)+\frac{4}{3}f=0.$$

As the equations by the Finite Difference Method for the symmetry points $(1,2)$ and $(2,1)$ do not differ, it is enough to consider equations for $(1,1)$, $(1,2)$, and $(2,2)$, say. Using the matrix representation

$$-9\frac{S'}{L^2}\begin{bmatrix}4 & -2 & 0\\ -1 & 4 & -1\\ 0 & -2 & 4\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}+\frac{f}{3}\begin{Bmatrix}2\\ 3\\ 4\end{Bmatrix}=0.$$

The solution can be obtained by Mathematica or (Gauss) elimination. First row operation to get an upper triangular matrix:

$$\begin{bmatrix}4 & -2 & 0\\ -4 & 16 & -4\\ 0 & -2 & 4\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}-\frac{fL^2}{27S'}\begin{Bmatrix}2\\ 12\\ 4\end{Bmatrix}=0 \quad \Rightarrow \quad \begin{bmatrix}4 & -2 & 0\\ 0 & 14 & -4\\ 0 & -2 & 4\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}-\frac{fL^2}{27S'}\begin{Bmatrix}2\\ 14\\ 4\end{Bmatrix}=0,$$

$$\begin{bmatrix}4 & -2 & 0\\ 0 & 14 & -4\\ 0 & -14 & 28\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}-\frac{fL^2}{27S'}\begin{Bmatrix}2\\ 14\\ 28\end{Bmatrix}=0 \quad \Rightarrow \quad \begin{bmatrix}4 & -2 & 0\\ 0 & 14 & -4\\ 0 & 0 & 24\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}-\frac{fL^2}{27S'}\begin{Bmatrix}2\\ 14\\ 42\end{Bmatrix}=0.$$

Then using the equations in reverse order (starting from the last one)

$$w_3=\frac{7}{108}\frac{fL^2}{S'} \quad \Rightarrow \quad w_2=\frac{fL^2}{27S'}+\frac{4}{14}w_3=\frac{6}{108}\frac{fL^2}{S'} \quad \Rightarrow \quad w_1=\frac{1}{2}(\frac{fL^2}{27S'}+w_2)=\frac{5}{108}\frac{fL^2}{S'}.$$

Altogether

$$\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}=\frac{fL^2}{S'}\frac{1}{108}\begin{Bmatrix}5\\ 6\\ 7\end{Bmatrix}. \quad \leftarrow$$