1 Basic Theory

The increasing operating and clock frequencies require transmission line theory to be considered more and more often!

1.1 Some practical transmission lines (waveguides)

Twin wire = Parallel two-wire
Twisted pair
Coaxial line
Slabline
Stripline
Microstrip line
Coplanar waveguide (CPW)
Optical fibre
Rectangular or circular waveguide
Electromagnetic waves in free space or dielectric
(Soundwaves and transmission line analogy)

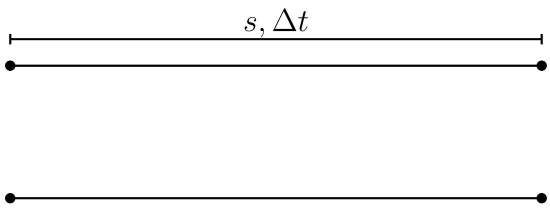

Fig. 1. A lossless transmission line; length s, delay Δt. No conductor resistances, no leakage between the conductors (ideal dielectric). Thus transmission line is exactly like a normal line.

Transmission line concept is needed if the line length s exceeds roughly $\lambda/50$ (or $\lambda/10 \ldots \lambda/100$), where lam(b)da λ is the wavelength of the signal. Also if the rise time or the length of a pulse is shorter than about $2\Delta t$, transmission line theory should be used.

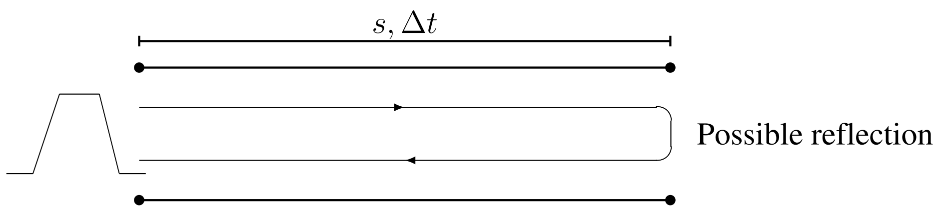

Fig. 2. A pulse moving on the line.

Transmission line theory is not needed if the lines are relatively short or if the pulses are long enough. Transmission line phenomena become obvious at higher frequencies, but you don't *necessarily* encounter these issues in everyday life.

1.2 Circuit theoretical model

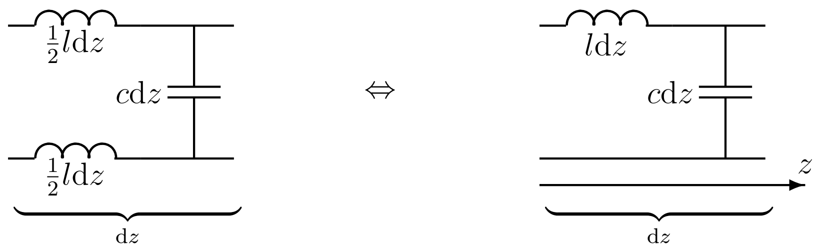

Fig. 3. A small section of a lossless transmission line (length $\mathrm{d}z$) modelled using lumped components, where l is inductance per unit length and c is capacitance per unit length (both distributed along the line); they are both dependent on the mechanical dimensions and the dielectric material.

1.3 Transmission line parameters

Characteristic impedance, wave velocity (= speed of light in the dielectric material), delay (s = line length in meters)

$$Z_{\mathrm{C}} = \sqrt{\frac{l}{c}} \tag{1}$$

$$v = \frac{1}{\sqrt{lc}} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{\mathrm{c}_0}{\sqrt{\mu_{\mathrm{r}}\varepsilon_{\mathrm{r}}}} \qquad \Delta t = \frac{s}{v} \tag{2}$$

1.3.1 Electrical material parameters

Permeability and relative permeability, permittivity (dielectric constant) and relative permittivity, speed of light in vacuum:

$$\mu = \mu_{\mathrm{r}}\mu_0 \qquad \mu_0 = 4\pi\,10^{-7}\ \mathrm{H/m} \tag{3}$$

$$\varepsilon = \varepsilon_{\mathrm{r}}\varepsilon_0 \qquad \varepsilon_0 = 8.854\,10^{-12}\ \mathrm{F/m} \tag{4}$$

$$c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \approx 3.0 \cdot 10^8\ \mathrm{m/s} \tag{5}$$

Note that $\mu_\mathrm{r} \approx 1$ for all non-magnetic (= non-ferromagnetic) materials (= paramagnetic and diamagnetic materials). Copper is diamagnetic $\mu_\mathrm{r} = 0.9999906 < 1$, but air is paramagnetic $\mu_\mathrm{r} = 1.00000037 > 1$. In vacuum, $\varepsilon_\mathrm{r} = 1$ and $\mu_\mathrm{r} = 1$. Thus ε_0 and μ_0 (*mu-subzero*) are the permittivity and permeability of vacuum, respectively.

1.3.2 Skin depth

The current density decreases exponentially as we go deeper inside a conductor. The phenomenon is caused by the so called *eddy currents*. Most of the current flows near the surface of the conductor inside a layer not deeper than δ (delta).

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \tag{6}$$

where sigma $\sigma = \frac{1}{\rho}$ is the conductivity (about 58 MS/m for copper, 1 S = 1 $\frac{1}{\Omega}$ and ρ (rho) is the resistivity. The conductor resistances increase very much at higher frequencies.

1.3.3 Coaxial line parameters

$$l = \frac{\mu}{2\pi} \ln \frac{r_\mathrm{outer}}{r_\mathrm{inner}} \qquad c = \frac{2\pi\varepsilon}{\ln \frac{r_\mathrm{outer}}{r_\mathrm{inner}}} \tag{7}$$

1.3.4 Twin wire ($d =$ distance of the conductors)

$$l = \frac{\mu}{\pi} \ln \frac{d}{r_\mathrm{conductor}} \qquad c = \frac{\pi\varepsilon}{\ln \frac{d}{r_\mathrm{conductor}}} \tag{8}$$

$$Z_\mathrm{C} = \sqrt{\frac{l}{c}} \tag{9}$$

2 DC Waves and Pulses

2.1 Interpretation of the characteristic impedance

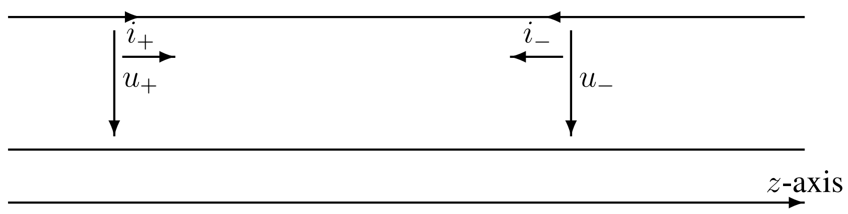

Fig. 4. Electromagnetic waves moving to the positive and negative direction of the horizontal axis (usually called the z-axis).

$$i_+ = \frac{u_+}{Z_C} \qquad\qquad i_- = \frac{u_-}{Z_C} \tag{10}$$

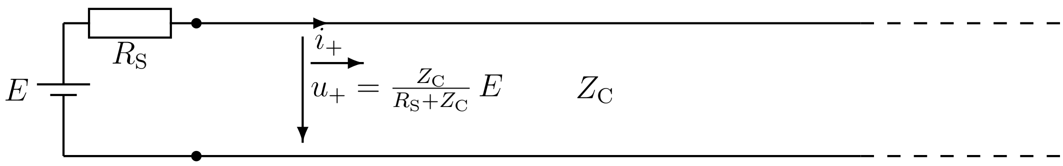

Fig. 5. A d.c. voltage connected on the line. The voltage divider rule is used. Voltage u_+ and current i_+ form the wave travelling to the line.

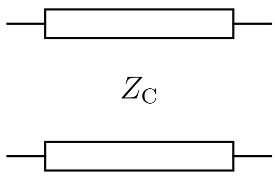

Fig. 6. The specific circuit element symbol of a transmission line (optional). Note! The bars are NOT resistances!!!!

2.2 Border conditions

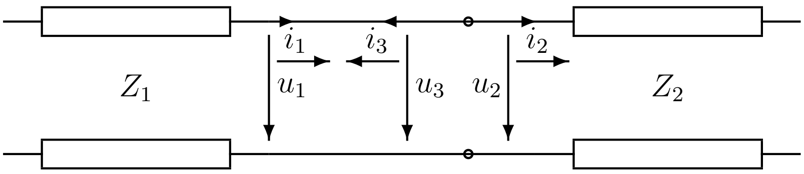

Fig. 7. Reflection at a discontinuity between two transmission lines. The incoming wave u_1 is partly reflected back (u_3) and only partly transmitted to the second line. Note that for the first line, $u_1 = u_+$, $i_1 = i_+$, $u_3 = u_-$ and $i_3 = i_-$.

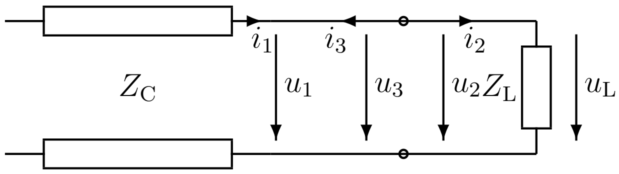

Fig. 8. Reflection at load Z_L. This case is mathematically exactly similar to the previous one.

$$i_1 - i_3 = i_2 \tag{11}$$

$$u_1 + u_3 = u_2 \tag{12}$$

The former equation can be interpreted as the Kirchhoff's current law and the latter one applies the superposition principle: u_1 and u_3 are separate waves that are added to get the total voltage. If u_1 and u_3 are both positive u_2 is greater than u_1. This may sound strange, but in fact, the power is NOT increased!

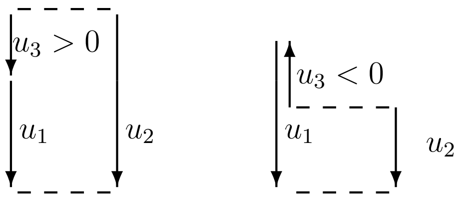

Fig. 9. Voltage superposition.

$$i_1 = \frac{u_1}{Z_1} \tag{13}$$

$$i_2 = \frac{u_2}{Z_2} \tag{14}$$

$$i_3 = \frac{u_3}{Z_1} \tag{15}$$

2.2.1 Reflection and transmission coefficients

$$\frac{u_1}{Z_1} - \frac{u_3}{Z_1} = \frac{u_1 + u_3}{Z_2} \tag{16}$$

$$\Rightarrow u_3 = \underbrace{\frac{Z_2 - Z_1}{Z_2 + Z_1}}_{\rho} u_1 \tag{17}$$

$$u_1 + \rho u_1 = \underbrace{(1+\rho)}_{\tau} u_1 = u_2 \tag{18}$$

$$\rho = \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{19}$$

$$\tau = 1 + \rho = \frac{2Z_2}{Z_2 + Z_1} \tag{20}$$

2.2.2 Propagating power

$$p_1 = \frac{u_1^2}{Z_1} \tag{21}$$

$$p_2 = \frac{u_2^2}{Z_2} \tag{22}$$

$$p_3 = \frac{u_3^2}{Z_1} \tag{23}$$

2.2.3 Voltage divider rule for a wave coming from a lumped source

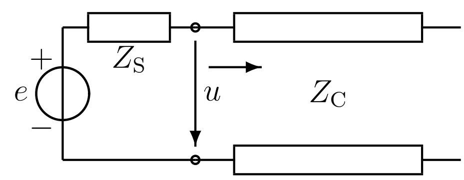

Fig. 10. The wave propagating to the line is found by using the voltage divider rule (as shown in Fig. 5).

$$u = u_+ = \frac{Z_C}{Z_S + Z_C} e \tag{24}$$

2.2.4 Shunt and series resistors

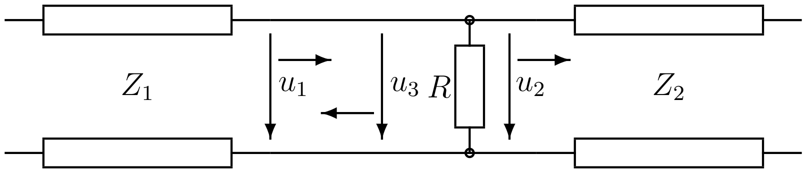

Fig. 11. Reflection at a shunt resistor.

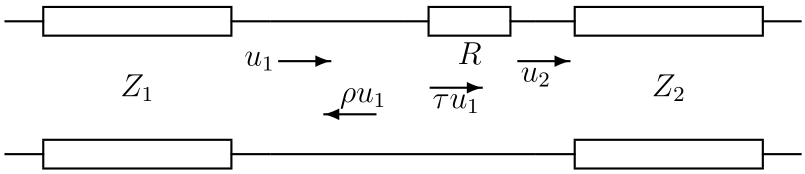

Fig. 12. Reflection at a series resistor.

2.2.5 Repeated reflections

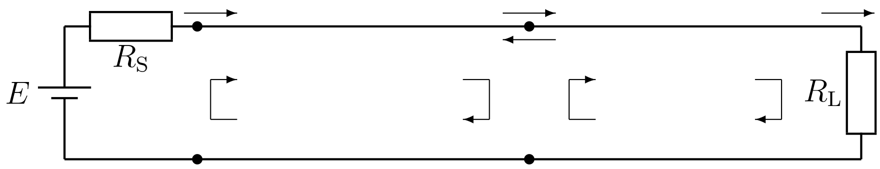

Fig. 13. The final voltages are a combination of infinite number of repeated reflections and transmissions.

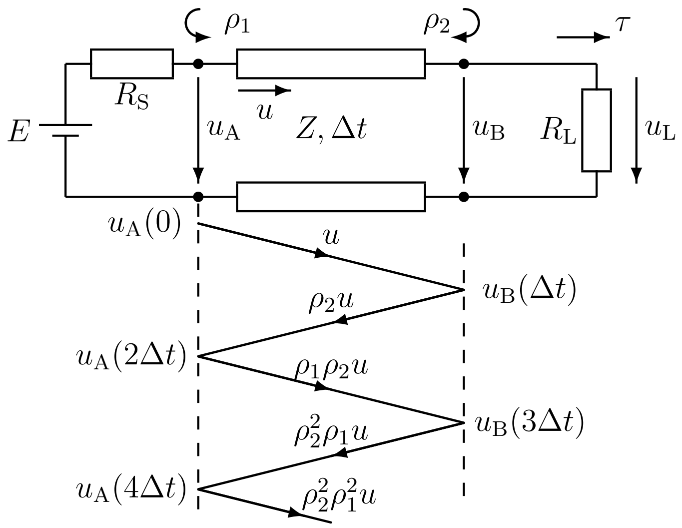

Fig. 14. Finding the voltages as a function of time. Voltage $u_A(0) = u_{A+}$ is first found by the voltage divider rule.

3 Sinusoidal signals

The repeated-reflections-technique would be very complicated in the sinusoidal case because of the phase difference of the waves. Usually one is mainly interested in the final continuous waveforms that are result of (infinite) number of repeated reflections.

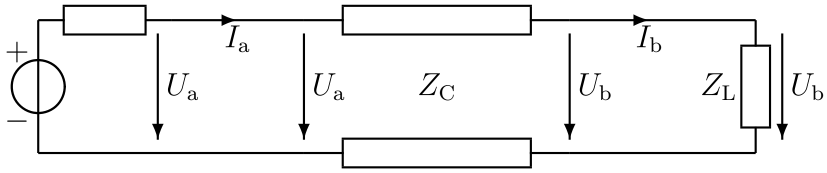

Fig 15. Voltage and current at both ends of the line (length s).

3.1 Voltage and current

The final effective voltage and current values can be found using the following equations:

$$U_\mathrm{a} = U_\mathrm{b}\cos\beta s + \mathrm{j}Z_\mathrm{C}I_\mathrm{b}\sin\beta s \tag{25}$$

$$I_\mathrm{a} = \mathrm{j}\frac{U_\mathrm{b}}{Z_\mathrm{C}}\sin\beta s + I_\mathrm{b}\cos\beta s \tag{26}$$

$$Z_\mathrm{C} = \sqrt{\frac{l}{c}} \tag{27}$$

$$v = \lambda f = \frac{s}{\Delta t} = \frac{\mathrm{c}_0}{\sqrt{\varepsilon_\mathrm{r}}} = \frac{1}{\sqrt{lc}} \tag{28}$$

$$\beta = \frac{\omega}{v} = \frac{2\pi}{\lambda} = \omega\sqrt{lc} \tag{29}$$

$$\Delta t = s/v = s\sqrt{\varepsilon_\mathrm{r}}/\mathrm{c}_0 \tag{30}$$

where β is the phase coefficient (wave number) in rad/m.

Equations in matrix form:

$$\begin{bmatrix} U_\mathrm{a} \\ I_\mathrm{a} \end{bmatrix} = \begin{bmatrix} \cos\beta s & \mathrm{j}Z_\mathrm{C}\sin\beta s \\ \mathrm{j}\frac{1}{Z_\mathrm{C}}\sin\beta s & \cos\beta s \end{bmatrix} \begin{bmatrix} U_\mathrm{b} \\ I_\mathrm{b} \end{bmatrix} \tag{31}$$

$$\begin{bmatrix} U_\mathrm{b} \\ I_\mathrm{b} \end{bmatrix} = \begin{bmatrix} \cos\beta s & -\mathrm{j}Z_\mathrm{C}\sin\beta s \\ -\mathrm{j}\frac{1}{Z_\mathrm{C}}\sin\beta s & \cos\beta s \end{bmatrix} \begin{bmatrix} U_\mathrm{a} \\ I_\mathrm{a} \end{bmatrix} \tag{32}$$

3.2 Impedance transformation

Load impedance seen through the line:

$$Z_\mathrm{a} = \frac{U_\mathrm{a}}{I_\mathrm{a}} = \frac{\overbrace{U_\mathrm{b}}^{Z_\mathrm{L} I_\mathrm{b}} \cos\beta s + \mathrm{j} Z_\mathrm{C} I_\mathrm{b} \sin\beta s}{\mathrm{j}\frac{U_\mathrm{b}}{Z_\mathrm{C}} \sin\beta s + I_\mathrm{b}\cos\beta s} = \frac{Z_\mathrm{L}\cos\beta s + \mathrm{j} Z_\mathrm{C}\sin\beta s}{\mathrm{j} Z_\mathrm{L}\sin\beta s + Z_\mathrm{C}\cos\beta s} Z_\mathrm{C} \tag{33}$$

$$\frac{Z_\mathrm{a}}{Z_\mathrm{C}} = \frac{\frac{Z_\mathrm{L}}{Z_\mathrm{C}} + \mathrm{j}\tan\beta s}{1 + \mathrm{j}\frac{Z_\mathrm{L}}{Z_\mathrm{C}}\tan\beta s} \tag{34}$$

The last equation is the definition of the Smith Chart. Smith Chart is a graphical tool for calculating impedance or its reflection coefficient along the line.

3.3 Lossy line

If the losses are taken into account:

$$U_\mathrm{a} = U_\mathrm{b}\cosh\gamma s + Z_\mathrm{C} I_\mathrm{b}\sinh\gamma s \tag{35}$$

$$I_\mathrm{a} = \frac{U_\mathrm{b}}{Z_\mathrm{C}}\sinh\gamma s + I_\mathrm{b}\cosh\gamma s \tag{36}$$

Propagation constant γ and characteristic impedance Z_C:

$$\gamma = \alpha + \mathrm{j}\beta = \sqrt{(r + \mathrm{j}\omega l)(g + \mathrm{j}\omega c)} \tag{37}$$

$$Z_\mathrm{C} = \sqrt{\frac{r + \mathrm{j}\omega l}{g + \mathrm{j}\omega c}} \tag{38}$$

Attenuation:

$$A = 20\lg \mathrm{e}^{\alpha s}\ \mathrm{dB} = 8{,}686\,\alpha s\ \mathrm{dB} \tag{39}$$

$$A = \alpha s\ \mathrm{Np} \tag{40}$$

3.4 Standing wave ratio, SWR

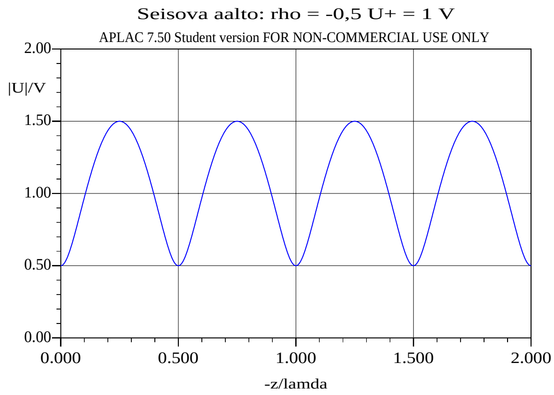

Fig. 16. A standing wave $U_+ = 1$ V, $U_- = \rho U_+$, $\rho = -0{,}5$.

$$\sigma = SWR = \frac{U_{\mathrm{MAX}}}{U_{\mathrm{MIN}}} = \frac{1+|\rho|}{1-|\rho|} \tag{41}$$

$$|\rho| = \frac{\sigma - 1}{\sigma + 1} = \frac{U_{\mathrm{MAX}} - U_{\mathrm{MIN}}}{U_{\mathrm{MAX}} + U_{\mathrm{MIN}}} \tag{42}$$

3.4.1 Maxima and minima in a standing wave

$$U_{\mathrm{MAX}} = (1+|\rho|)U_+ \qquad U_{\mathrm{MIN}} = (1-|\rho|)U_+ \tag{43}$$

$$I_{\mathrm{MIN}} = (1-|\rho|)\frac{U_+}{Z_{\mathrm{C}}} \qquad I_{\mathrm{MAX}} = (1+|\rho|)\frac{U_+}{Z_{\mathrm{C}}} \tag{44}$$

4 Smith Chart

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