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Additive value functions

Attribute-specific value functions, conditions required, attribute weights, elicitation

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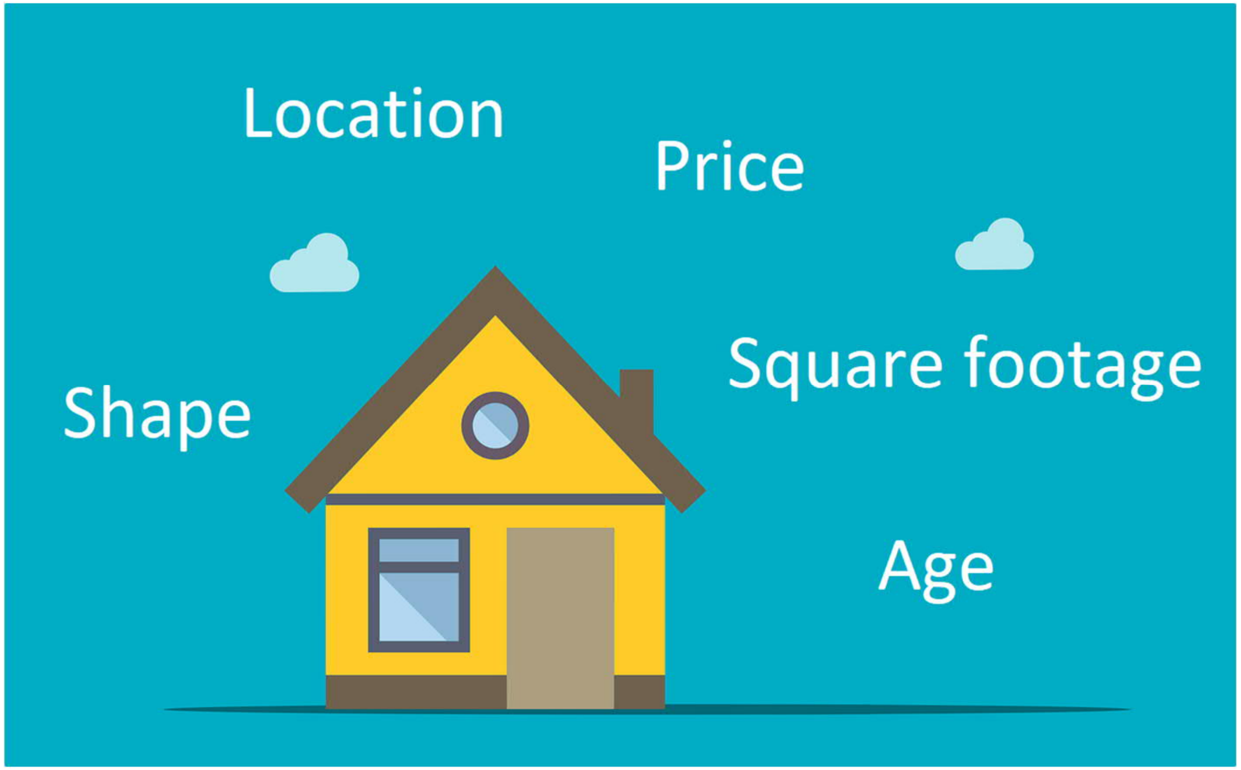
Example: Buying a house



	Price	Size	Shape	Location	Age (years)
House A	500 000 €	60 m^2	Decent	Decent	10
House B	650 000 €	50 m^2	Good	Excellent	40
House C	350 000 €	80 m^2	Excellent	Poor	0

Content of the lecture

1. **Preferences between alternatives**
2. **Attribute-specific value functions**
3. **Additive value functions**
4. **Attribute weights**
5. **Elicitation of value functions and attribute weights**

I) Preferences between alternatives

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Preferences between alternatives with one objective under certainty

- **With one objective, each alternative constitutes of a single attribute**
- **Attributes can be**
  - Ordinal, e.g., “good”, “decent”, “poor”
  - Cardinal, e.g., 350 000 €, 500 000 €, 650 000 €
- **Attributes have various levels**
  - Here these levels are considered to be certain, i.e., no stochasticity is involved

Conditions for preferences

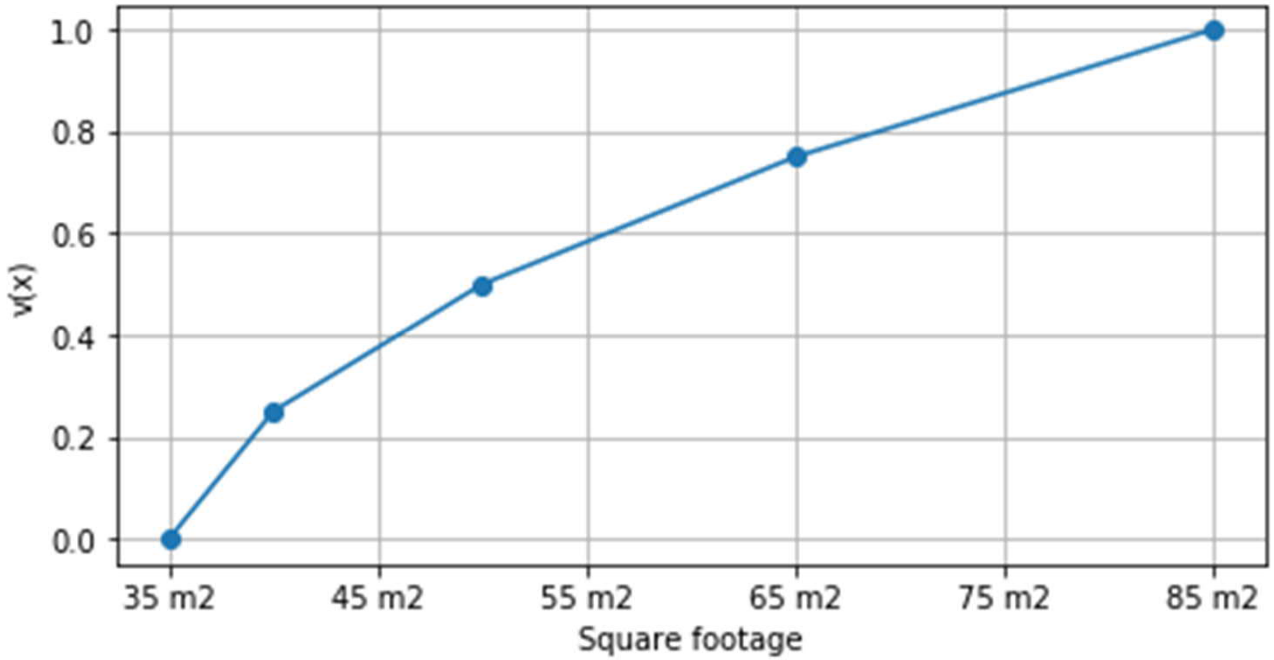
- **Preferences for alternatives:**
  - $a \succ b$: a is preferred over b
  - $a \sim b$: indifference between a and b
- **Preferences for alternatives are**
  - Complete, if there is a preference between any pair of alternatives:
    - $(a \succ b)$, $(b \succ a)$ or $(a \sim b)$, for any a and b
  - Transitive, if for any three alternatives holds
    - $(a \succ b)$ & $(b \succ c)$ => $(a \succ c)$

II) Attribute-specific value functions

Attribute-specific value functions

- **Function that evaluates preferences between alternatives is called *value function***
- **For a valid (ordinal) value function, it holds that:**
  - $v(a) > v(b)$ if and only if $a \succ b$
  - $v(a) = v(b)$ if and only if $a \sim b$
- **For a valid <u>cardinal value function</u>, it also holds that:**
  - $[v(b) - v(a)] > [v(d) - v(c)] \Leftrightarrow (a \rightarrow b) \succ (c \rightarrow d)$
  - Where the preferences for transitions must also be
    - Complete
    - Transitive

Attribute-specific value functions



III) Additive value functions

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Additive value function

- **Value function for multi-attribute alternatives**
- **Alternative is represented as a vector of attribute levels**
  - $x = (x_1, \dots, x_n)$
  - For example, *x* = (500 000 €, 60 m², Decent, Decent, 10 years)
- **Additive value function adds together the weighted and normalized attribute-specific value function scores $v_i(x_i)$:**
  - $V(x) = V(x_1, \dots, x_n) = \sum_{i=1}^{n} w_i \, v_i(x_i)$,

    where w_i is the weight of the attribute x_i,

    and $\sum_{i=1}^{n} w_i = 1$ and $v_i(x_i) \in [0,1]$.

Requirements for attributes

- **There exists an additive value function, if preferences for alternatives and transitions between them are *complete* and *transitive*, and all attributes are**
  - Mutually preferentially independent
  - Difference independent

Mutual preferential independence

- **Attributes A are *mutually preferentially independent*, if any subset $X \subset A$ of them is *preferentially independent* of the other attributes *Y* = $A \backslash X$.**
  - Attribute X is *preferentially independent* of the other attributes Y, if for all $x, \bar{x} \in$ X:
    - $(\bar{x}, y') \succ (x, y') \Rightarrow (\bar{x}, y) \succ (x, y)$, for all $y, y' \in$ Y.
    - For example, if "Good" location is preferred to "Decent" location, the price or the square footage of the apartment does not affect this preference.

Difference independence

- **Attribute *X* is difference independent of the other attributes *Y*, if for all $\boldsymbol{x}, \overline{\boldsymbol{x}} \in \mathbf{X}$:**
  - $(x, y') \rightarrow (\bar{x}, y') \sim (x, y) \rightarrow (\bar{x}, y)$, for all $y, y' \in Y$.
  - The level of preference for a transition from x to $\bar{x}$ does not depend on the other attribute levels, as long as they stay the same.
  - For example, a change from “Decent” location to “Good” location is equally preferred whether the price is 350 000 € or 500 000 € if the price stays at the same level in the change.

IV) Attribute weights

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Attribute weights

- **Weights do not reflect the absolute "importance" of each attribute, but the trade-offs between them**
  - For example, attribute weights do not answer to the question:
    - *"Which is more important in buying a house, its price or its location?"*
  - But rather to the question:
    - *"How much is the DM willing to pay in order to improve the location?"*
- **Weights are also dependent on the attribute intervals!**

$$V(x) = \sum_{i=1}^{n} w_i \, v_i(x_i)$$

V) Elicitation of value functions and attribute weights

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Elicitation of value functions

- **Value functions are normalized between [0,1]**
  - $v(x^-) = 0$ is the least preferred level of attribute x
  - $v(x^+) = 1$ is the most preferred level of attribute x
- **Value functions are monotonically increasing**
  - Preferences are not guaranteed to be monotonic!
- **Value functions are elicited by questioning the DM either directly or indirectly**
- **Errors can occur, and thus it is important to perform consistency checks**

Elicitation of value functions

- **The bisection method**
  - Determine the least and the most preferred levels x^- and x^+
  - Divide the attribute level range in two parts so that the DM is indifferent between the transitions
    - $(x^- \rightarrow x_{0.5}) \sim (x_{0.5} \rightarrow x^+)$
  - Divide the resulting ranges in halves again with the same idea
    - $(x^- \rightarrow x_{0.25}) \sim (x_{0.25} \rightarrow x_{0.5})$
    - $(x_{0.5} \rightarrow x_{0.75}) \sim (x_{0.75} \rightarrow x^+)$
  - Interpolate between the elicited points

Elicitation of attribute weights

- **The trade-off method**
  - Based on comparing alternatives that differ in two attributes but are considered to be equally attractive
    - For example, $(x_1, x_2, x_3, x_4) \sim (x_1, \bar{x}_2, x_3, \bar{x}_4)$
      => $w_1 v_1(x_1) + w_2 v_2(x_2) + w_3 v_3(x_3) + w_4 v_4(x_4) =$
      $w_1 v_1(x_1) + w_2 v_2(\bar{x}_2) + w_3 v_3(x_3) + w_4 v_4(\bar{x}_4)$
  - This results in *n-1* equations and combining them with the fact that the weights sum up to one, we can solve each respective attribute weight
  - The attribute-specific value functions must be known

Example

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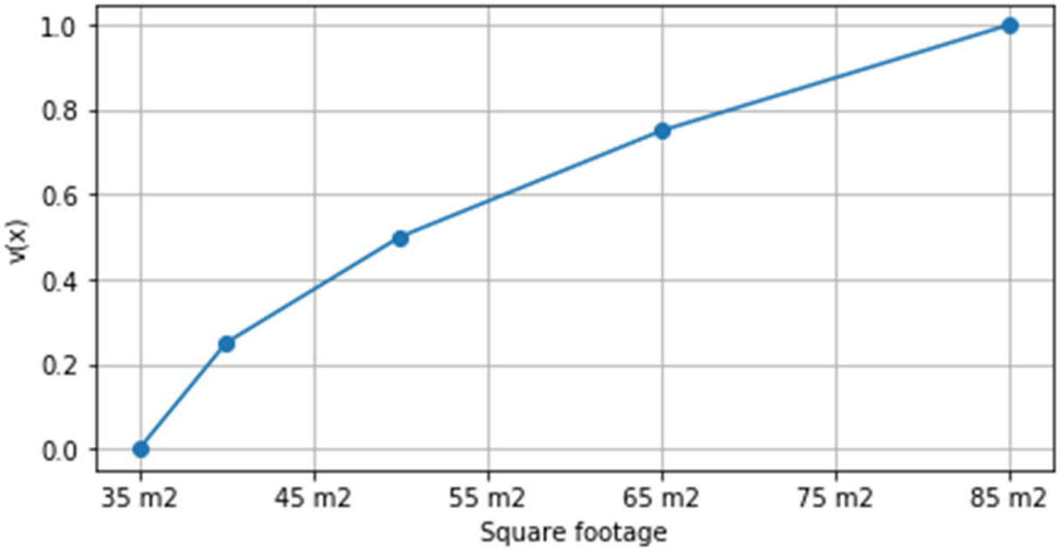
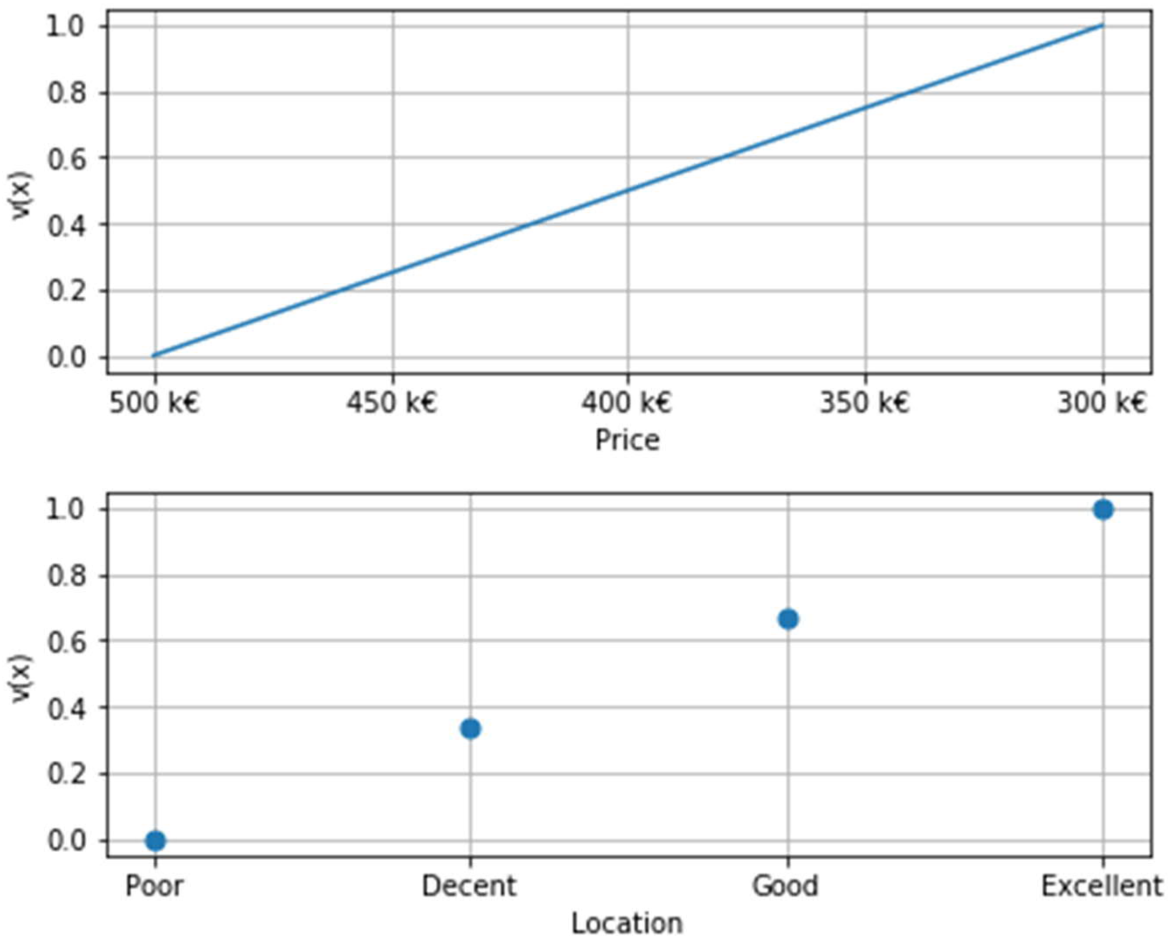
	Price	Square footage	Location
A	300 k€	35 m²	Good
B	350 k€	70 m²	Poor
C	350 k€	45 m²	Excellent
D	450 k€	60 m²	Good
E	500 k€	85 m²	Decent

Let's assume that for the price and location the attribute-specific value functions are linear, but the value function of the square footage should be elicited considering the preferences.

We will use the bisection method for the value function and the trade-off method for the attribute weights.

- **The least and the most preferred levels:**
  - v(35) = 0
  - v(85) = 1
- **Finding the bisection points:**
  - (35 → 50) ~ (50 → 85)
    - => v(50) = 0.5
  - (50 → 65) ~ (65 → 85)
    - => v(65) = 0.75
  - (35 → 40) ~ (40 → 50)
    - => v(40) = 0.25




1.0
0.8
0.6
0.4
0.2
0.0
v(x)
500 k€
450 k€
400 k€
350 k€
300 k€
Price
Poor
Decent
Good
Excellent
Location

- **Trade-off comparisons (due to the preferential independence the remaining attributes can be marked with *)**
  - (300 k€, 40 m², *) ~ (340 k€, 65 m², *)
  - (300 k€, *, Poor) ~ (420 k€, *, Excellent)
- **From the definition of additive value function:**
  1) $w_1 v_1$(300 k€) + $w_2 v_2$(40 m²) = $w_1 v_1$(340 k€) + $w_2 v_2$(65 m²)
  2) $w_1 v_1$(300 k€) + $w_3 v_3$(Poor) = $w_1 v_1$(420 k€) + $w_3 v_3$(Excellent)
  3) $w_1 + w_2 + w_3 = 1$
- **We get the attribute weights:**
  - $w_1 = 0.5$
  - $w_2 = 0.2$
  - $w_3 = 0.3$

Additive value function on the example

	Price	Square Footage	Location	v1(x)	v2(x)	v3(x)	V(X)
A	300 k€	35 m²	Good	1.00	0.00	0.67	0.70
B	350 k€	70 m²	Poor	0.75	0.81	0.00	0.54
C	350 k€	45 m²	Excellent	0.75	0.38	1.00	0.75
D	450 k€	60 m²	Good	0.25	0.67	0.67	0.46
E	500 k€	85 m²	Decent	0.00	1.00	0.33	0.30

$$V(x) = \sum_{i=1}^{n} w_i \, v_i(x_i)$$

w1	w2	w3
0.5	0.2	0.3

Summary

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Summary

- **Additive values functions provide a way to evaluate subjective preferences between multi-attribute alternatives in a systematic and rational manner**
- **Additive value functions can be used for evaluating multi-attribute alternatives if**
  - preferences for alternatives and transitions between them are complete and transitive,
  - and if all attributes are
    - *mutually preferentially independent*
    - *difference independent*

Summary

- **Attribute-specific value functions describe the DM's preferences between the attribute levels**
  - Ordinal value function describes the order of the levels
  - Cardinal value function describes also the order of the transitions
  - Value function can be elicited e.g., with the bisection method
- **Weights do not reflect the absolute "importance" of each attribute, but the trade-offs between them**
  - Weights can be elicited e.g., with the trade-off method

References

1. Eisenführ et al. (2010). Rational Decision Making. Springer-Verlag Berlin Heidelberg. Pages: 107-142, 151-154.
2. MS-E2134 Lecture material

Homework

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Homework (1/2)

Consider the data presented in the example on the slide 23. Assume that for the DM, the value functions for the square footage and the location are linear between the least and the most preferred levels and that the DM prefers larger house to smaller house and better location to worse location. However, the value function of the price should be elicited with the bisection method. The DM states that they prefer cheaper house compared to expensive house and that they are indifferent between (500 k€ → 450 k€) and (450 k€ → 300 k€).

1. Derive and plot the attribute-specific value function of the price as the function is assumed to be linear around the bisection point.

Homework (2/2)

The DM gives the following trade-offs that they feel indifferent about:

- (*, 85 m^2, Poor) ~ (*, 75 m^2, Excellent)
- (450 k€, 65 m^2, *) ~ (500 k€, 85 m^2, *)

2. Derive the normalized attribute weights and calculate the corresponding additive value function scores for the alternatives. Which alternative would be the best for the DM?

Send the answers by 1.10.2021 at 09:00 to *aleksi.avela@aalto.fi*