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Nonadditive portfolio value functions

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Overview

1. Running example
2. Recap: Additive value functions
3. Nonadditive portfolio value functions
4. Eliciting nonadditive value functions
5. How to solve the optimal portfolio
6. Summary

Running example: Ecological conservation site selection (Liesiö (2014))

- **Goal: purchase privately-owned forest sites for conservation**
- **50 conservation sites (m=50) evaluated based on 5 criteria (n=5)**
- **Maximize conservation value of a site portfolio with limited budget**

Table 1 Criteria and Measurement Scales

Table from Liesiö (2014)

i	Criterion name	Measurement unit	x_i^0	x_i^*	X_i
1	Area	ha	0.5	5	[0.5, 5]
2	Old broad-leaved trees	m^3	0	200	[0, 200]
3	Natural water economy	Verbal	None	Excellent	{none, poor, good, excellent}
4	Endangered species	Number	0	100	$\{0, 1, \ldots, 100\}$
5	Closest natural reserve	km	50	0	[0, 50]

Recap: Additive value functions[1,2]

- **Additive value function adds weighted (w_i) and normalized attribute-specific values (v_i) together**

$$V(x) = \sum_{i=1}^{n} w_i v_i(x_i)$$

  - Preferences are complete and transitive
  - Attributes are mutually preferentially independent and difference independent

Recap: Additive-linear portfolio value function[1,2]

- **Portfolio value calculated by summing weighted criterion-specific value functions for each n attributes in a portfolio of m projects**
  - Portfolio decision analysis models often rely on this
  - Criterion-specific project value function v_i
  - Criterion-specific portfolio value function V_i

$$V(x) = \sum_{i=1}^{n} V_i(x_{Ji}),$$

$$V_i(x_{Ji}) = w_i \sum_{j=1}^{m} v_i(x_{ji}), i = 1, \dots, n$$

Recap: Assumptions[1,2]

1. **Preferences are project symmetric**
   - Portfolio performances that are equal up to permutation of rows are equally preferred
2. **Each attribute is WDI**
   - Preference order of changes in attribute levels remains the same for any levels of other attributes
3. **Each set of attributes measuring criterion-specific performance is WDI**
   - Each criterion can be used as a meaningful measure of portfolio performance by examining project performances

Assumptions 1-3 will hold throughout this presentation

Assumption 4?

4. Each set of attributes measuring a single project is DI

- Any change in performance levels of a single project remains equally preferred even if performances of the other projects in the portfolio are varied
- Necessary for additive-linear function

→ **Adding a site into the portfolio results in the same value increase independent of the portfolio composition**

Assumption 4 – Conservation site (counter)example

- **“Adding a site into the portfolio results in the same value increase independent of the portfolio composition”**
  - In the conservation site selection, it can be that criterion i=3 “natural water economy” is more important in an empty portfolio than i=4 “endangered species”
  - DM would rather select site (0.5ha, 0m3, exc, 0, 10km) than (0.5ha, 0m3, none, 100, 10km) when the portfolio doesn’t contain any other sites
  - If portfolio contains many sites with excellent natural water economy, first option could be valued lower

→ Assumption 4 discarded

→ Assumption 5

- Each attribute X_{ji} is conditionally DI of other attributes in the same project $X_{\bar{j}i}$ given a fixed level of the remaining attributes $X_{j\bar{i}}$

$$\begin{pmatrix} 100 & 1 & 1 \\ 5 & 5 & 5 \\ 7 & 7 & 7 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 1 & 1 \\ 5 & 5 & 5 \\ 7 & 7 & 7 \end{pmatrix} \sim \begin{pmatrix} 100 & 10 & 5 \\ 5 & 5 & 5 \\ 7 & 7 & 7 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 10 & 5 \\ 5 & 5 & 5 \\ 7 & 7 & 7 \end{pmatrix}$$

- Changes in criterion-specific performance of a project remain equally preferred when other project’s performances are fixed

→ **Nonadditive value function!**

Nonadditive value functions: Additive-multilinear value function

- **Preferences satisfy assumptions 1-3 and 5**

$$\boxed{\boxed{V(x)}} = \sum_{i=1}^{n} V_i(x_{Ji})$$

$$V_i(x_{Ji}) = \sum_{J' \subseteq J} w_i(|J'|) \prod_{j \in J'} v_i(x_{ji}) \prod_{j \notin J'} (1 - v_i(x_{ji}))$$

- **Portfolio value $V(x)$ is the sum of the criterion-specific value functions V_i (just like with the additive-linear case)**

Additive-multilinear value function

$$V(\boldsymbol{x}) = \sum_{i=1}^{n} V_i(\boldsymbol{x}_{Ji})$$

$$\boxed{\boxed{V_i(\boldsymbol{x}_{Ji})}} = \sum_{J' \subseteq J} w_i(|J'|) \prod_{j \in J'} v_i(x_{ji}) \prod_{j \notin J'} (1 - v_i(x_{ji}))$$

- **Each criterion-specific value function is a symmetric strictly-increasing multilinear function of the criterion-specific project values** $v_i(x_{1i}), \dots, v_i(x_{mi})$

Additive-multilinear value function

$$V(x) = \sum_{i=1}^{n} V_i(x_{Ji})$$

$$V_i(x_{Ji}) = \sum_{J' \subseteq J} \boxed{w_i(|J'|)} \prod_{j \in J'} v_i(x_{ji}) \prod_{j \notin J'} (1 - v_i(x_{ji}))$$

- **Strictly increasing weighting function $w_i(1), .., w_i(m), w_i(0) = 0$**
  - $w_i(k)$ corresponds to the criterion-specific value of a portfolio that has k projects with indices $J' \subseteq J$ on the most preferred level, remaining $m - k$ projects on the least preferred level
- **Whiteboard example with portfolio $J = \{A, B\}$**

Numerical example – criterion-specific values

Project	Criterion-specific project values $v_i(x_j)$
A	0.4
B	0.6

Calculate $V_1(x_{J1})$ with $J_1 = \{A\}$ and $J_2 = \{A, B\}$ with $w_1(1) = 0.3$ and $w_1(2) = 0.7$

- $V_1\left(x_{J_1}\right) = w_1(1)v_1(x_{A1}) = 0.4 * 0.3 = 0.12$
- $V_1\left(x_{J_2}\right) = w_1(1)v_1(x_{A1})\left(1 - v_1(x_{B1})\right) + w_1(1)v_1(x_{B1})\left(1 - v_1(x_{A1})\right) + w_1(2)v_1(x_{A1})v_1(x_{B1}) = 0.3 * 0.4 * (1 - 0.6) + 0.3 * 0.6 * (1 - 0.4) + 0.7 * 0.4 * 0.6 = 0.324$

Some notation

$$\langle k_1, y; k_2, y' \rangle = (\underbrace{y, \ldots, y}_{\substack{k_1 \\ \text{elements}}}, \underbrace{y', \ldots, y'}_{\substack{k_2 \\ \text{elements}}}, \underbrace{x_i^0, \ldots, x_i^0}_{\substack{m-k_1-k_2 \\ \text{elements}}})^T \in X_{Ji}.$$

- **Portfolio of m projects with k_1 projects at performance y and k_2 projects at performance level y'**
- **By definition, $w_i(k) = V_i(\langle k, x^* \rangle)$**

Eliciting values & weights

- **v_i can be elicited with standard techniques**
- **Specification of V_i requires defining the weighting function values $w_i(1), \ldots, w_i(m)$ and $w_i(k)$ = V_i(⟨k,x*⟩)**
- **Weights with linear constraints:**
  - Ask DM to adjust level y until portfolios with $\langle k, x^* \rangle$ and $\langle k - 1, x^*; 2, y \rangle$ are equally preferred
  - Repeat for each $k \in \{1, \ldots, m-1\}$
  - Get $m - 1$ linear equations for m variables → determines the weighting function up to a positive constant

Eliciting weights – linear constraints method[1]

- **"Define volume y of old broad-leaved trees between $[x^0, x^*] = [0, 200]$ such that having nine sites with 200m^3 and two sites with y m^3 of broad-leaved trees is equally preferred to having ten sites with 200m^3 of broad-leaved trees"**

$$
\begin{aligned}
V_i(\langle k, x^* \rangle) &= V_i(\langle k-1, x^*; 2, y \rangle) \\
\Leftrightarrow w_i(k) &= w_i(k-1)(1 - v_i(y))^2 \\
&\quad + 2w_i(k)(1 - v_i(y))v_i(y) \\
&\quad + w_i(k+1)v_i(y)^2 \\
\Leftrightarrow w_i(k+1) - w_i(k) &= \left(\frac{1 - v_i(y)}{v_i(y)}\right)^2 (w_i(k) - w_i(k-1))
\end{aligned}
$$

Eliciting weights – DSS method[1]

- **Difference standard sequence**
  - Define unit stimulus $w_i(k_0)$, example: change from a portfolio with no water economy (x^0) to a portfolio of k_0 = 10 sites with excellent water economy (x^*)
  - Ask DM to define the number of sites k_1 s.t. the change from k_0(=10) to k_1 sites with excellent natural water economy is equally preferred to the change from zero to k_0(=10) such sites.
  - We get a sequence of portfolio performances where each change $\langle k_l, exc.\rangle \leftarrow \langle k_{l-1}, exc.\rangle$ is equally preferred

Eliciting weights – DSS[1]

- **Equality then used to define weighting function value for each k_l**

$$V_3(\langle k_l, exc\rangle) - V_3(\langle k_{l-1}, exc\rangle)$$
$$= V_3(k_0, exc) - V_3(0, exc)$$
$$\Leftrightarrow w_3(k_l) - w_3(k_{l-1}) = w_3(k_0)$$

- **Remaining values with linear interpolation**
  - Absolute weighting function value can be fixed with tradeoff techniques, swing weighting etc

Figure 3 **Large Dots Correspond to the Weighting Function Values Obtained from the Difference Standard Sequence $k_l = 0, 10, 13, 15, 18, 23, 50$ and Small Dots to the Interpolated Values**

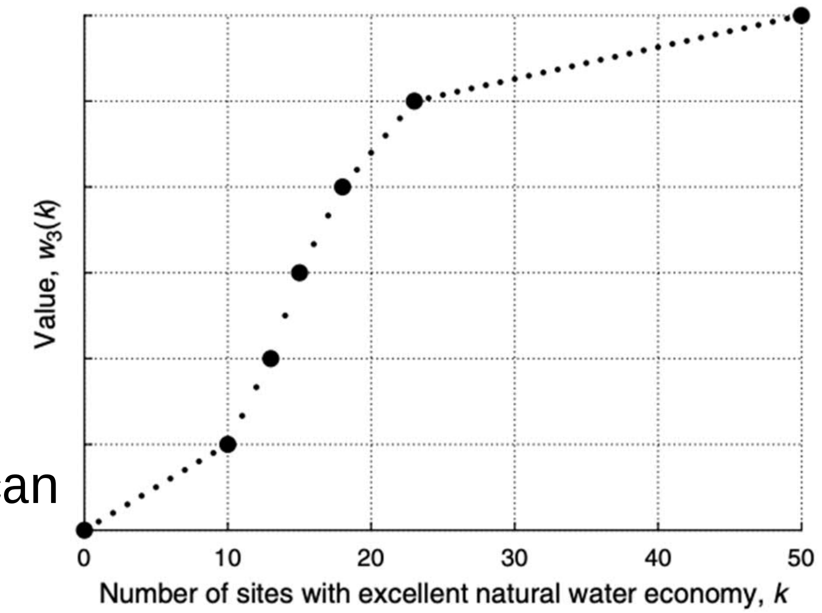
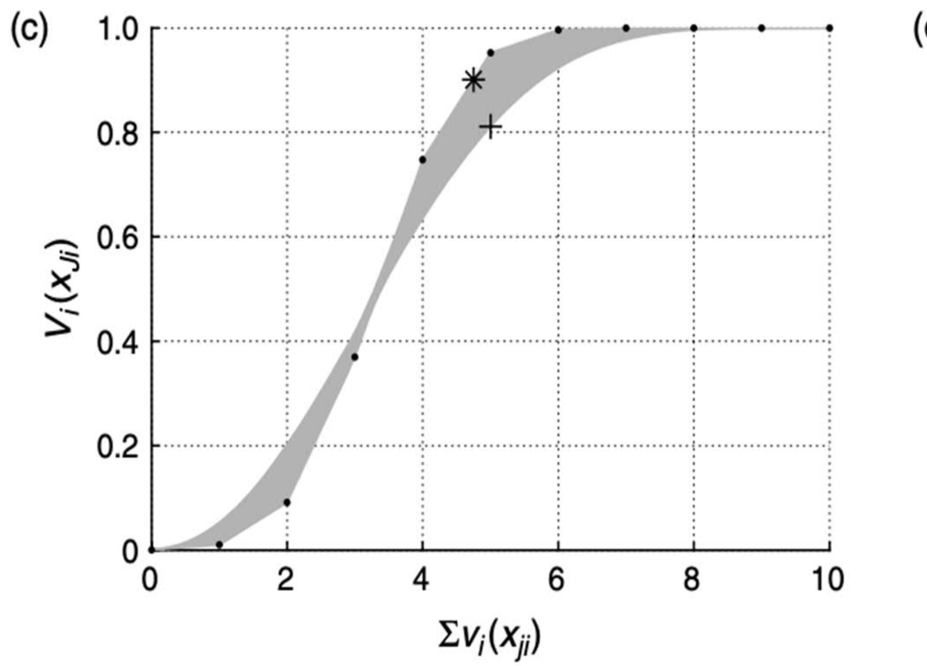


Figure from Liesiö (2014)

Comparison: Sum of project scores vs. Additive-multilinear

- **Additive-multilinear criterion-specific V_i and sum of criterion specific project values with $m = 10$**
- **Dots show weighting function values from $1, .., 10$**
- **Gray area is the set of points obtained when portfolio performance is varied through its entire domain**

Figure from Liesiö (2014)



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1. $\langle 5, \boldsymbol{x}^* \rangle$, $\boldsymbol{w_i}(\mathbf{5}) = \mathbf{0.95}$: *sum of project scores = 5*

- $\boldsymbol{V_i}(\langle \mathbf{5}, \boldsymbol{x}^* \rangle) = \mathbf{0.01} * \mathbf{1} * (\mathbf{1} - \mathbf{1})^{\mathbf{4}} + \cdots + \mathbf{0.95} * \mathbf{1} = \mathbf{0.95}$

(o)

2. $\langle \mathbf{10}, \boldsymbol{y} \rangle$, $\boldsymbol{v_i}(\boldsymbol{y}) = \mathbf{0.5}$: *sum of project scores = 5*

(+)

- $\boldsymbol{V_i}(\langle \mathbf{10}, \boldsymbol{y} \rangle) = \cdots = \mathbf{0.81}$

3. $\langle \mathbf{4}, \boldsymbol{x}^*; \mathbf{1}, \boldsymbol{y}' \rangle$, $v_i(y') = 0.75$, $\boldsymbol{w_i}(\mathbf{4}) \approx \mathbf{0.75}$: *sum of project scores = 4.75*

- $\boldsymbol{V_i}(\langle \mathbf{4}, \boldsymbol{x}^*; \mathbf{1}, \boldsymbol{y}' \rangle) = \ldots + \mathbf{0.75} * \mathbf{1}^{\mathbf{4}} * (\mathbf{1} - \mathbf{0.75}) + \mathbf{0.95} * \mathbf{1}^{\mathbf{4}} * \mathbf{0.75} \approx \mathbf{0.90}$

(*)

Additive-multilinear value function isn't always increasing in the sum of scores and can't be accurately represented with a nonlinear function

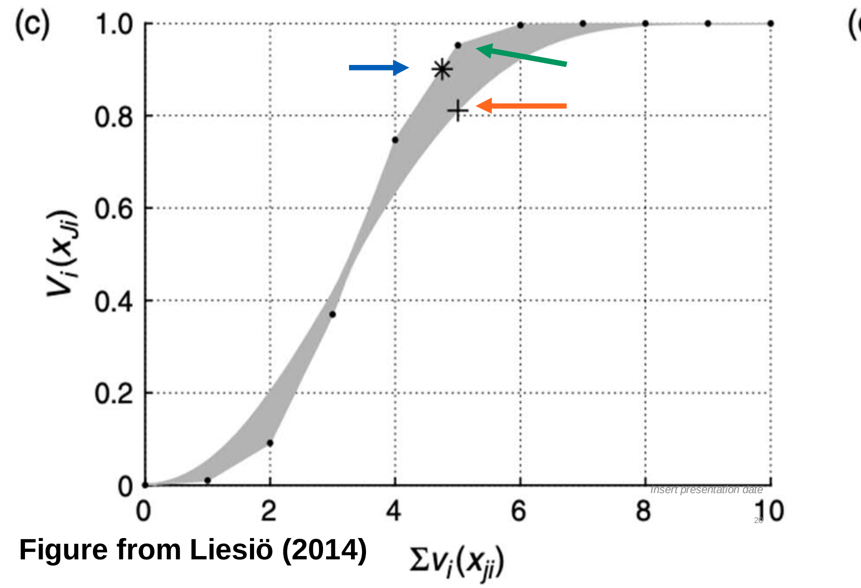
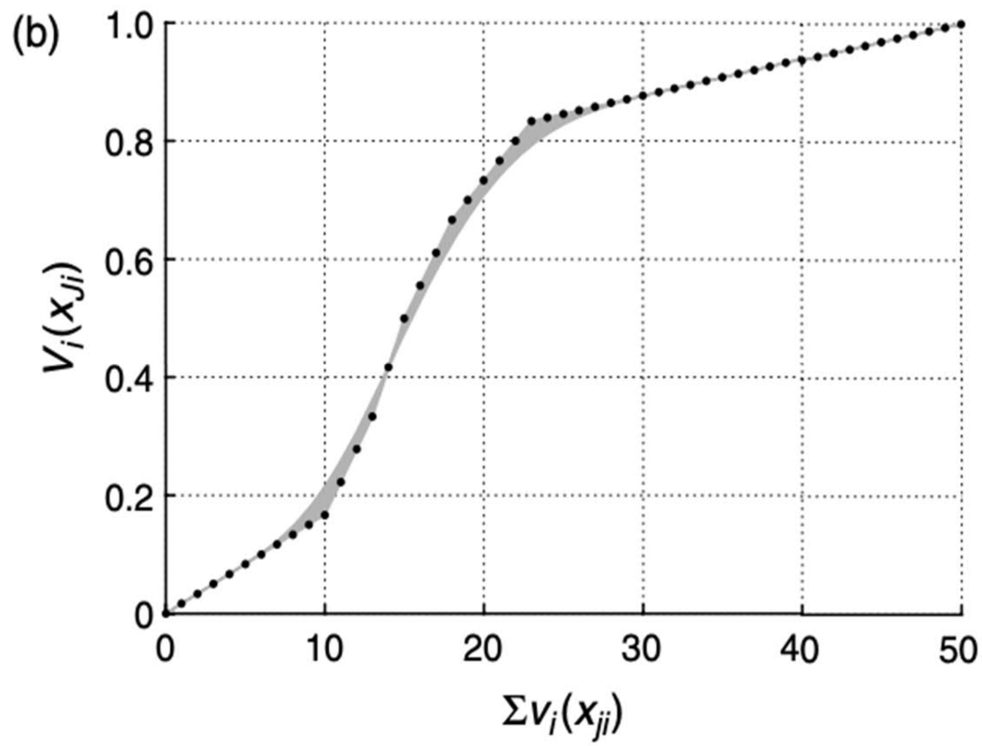


Figure from Liesiö (2014)

Sum of project scores vs. Additive-multilinear

- **Linear weighting function results in a linear portfolio value function**
- **In some cases, sum of project scores can give a good approximation of V_i**

Figure from Liesiö (2014)



Additive-multilinear portfolio value function: Summary

- **Satisfies assumptions 1-3, 5**
- **More general case than additive-linear**
  - Additive-linear can sometimes be a good approximation
  - Shares many steps with applying additive-linear portfolio models (defining the problem and project scoring), but value of adding a project into the portfolio doesn't have to be constant
- **Number of parameters increases linearly in number of attributes**

The multilinear portfolio value function[1]

- **If preferences only satisfy assumptions 1-3, general form:**

$$V(x) = \sum_{I' \subseteq I} \lambda(I') \prod_{i \in I'} \frac{V_i(x_{Ji})}{\lambda(\{i\})} \prod_{i \notin I'} \left(1 - \frac{V_i(x_{Ji})}{\lambda(\{i\})}\right).$$

- V_i **can be elicited as described previously, then the values of** λ **can be obtained by examining preferences over the performances of a single project**
- **Elicitation becomes exponentially more computationally expensive in the number of criteria**

Optimization models for maximising portfolio value[1]

- **Optimal portfolio problem can be formulated as a nonlinear integer programming problem (maximize portfolio value)**
- **$\boldsymbol{m}$ < 100 & Additive-multilinear V:**
  - Enumeration algorithm (depth-first binary tree search) presented (see Liesiö 2014)
  - Number of solutions to enumerate reduced by testing if a specific branch will only contain suboptimal or infeasible solutions
  - Conservation site example with $m = 50$ took less than 2 seconds to solve on a standard computer

Optimization models for maximising portfolio value[1,3]

- **Larger problems: Approximate with a MILP model**
  - First choose a piecewise linear mapping $\tilde{V}_i: [0, m] \rightarrow [0,1]$ for each criterion $i \epsilon I$ → Approximate optimization problem
  - This can be formulated as a MILP problem and solved
  - Conservation site took less than a second to solve

Optimization models for maximising portfolio value[1,3]

- **Approximate MILP formulation**
  - $\tilde{V}_i$ linear piecewise approximation of V_i
  - θ, ψ are nonnegative scalars for linearisation
  - $Az \leq B$ is budget constraint
  - z_j = 1 if project j is chosen
  - χ_d^i points in interval [0,m] for approximation $\tilde{V}_i$

$$\max_{\substack{z \in \{0,1\}^m \\ \theta^i, \psi^i, i \in I}} \left\{ \sum_{i=1}^{n} \sum_{d=1}^{l_i} \theta_d^i \tilde{V}_i(\chi_d^i) \right\}$$

$$Az \leq B,$$

$$\sum_{j=1}^{n} [z_j v_i(x_{ji}) + (1 - z_j) v_i(\underline{x}_i)] = \sum_{d=1}^{l_i} \theta_d^i \chi_d^i \quad \forall i \in I,$$

$$\sum_{d=1}^{l_i} \theta_d^i = 1 \quad \forall i \in I,$$

$$\sum_{d=1}^{l_i - 1} \psi_d^i = 1 \quad \forall i \in I,$$

$$\theta_1^i \leq \psi_1^i \quad \forall i \in I,$$

$$\theta_d^i \leq \psi_{d-1}^i + \psi_d^i \quad \forall d \in \{2, \ldots, l_i - 1\}, i \in I,$$

$$\theta_{l_i}^i \leq \psi_{l_i - 1}^i \quad \forall i \in I,$$

$$\theta^i \in [0,1]^{l_i}, \quad \psi^i \in \{0,1\}^{l_i - 1}, \quad \forall i \in I.$$

Summary: Conservation site

- **Additive-multilinear portfolio value function used**
  - Assumptions can be relaxed, project synergy allowed, more general than additive value functions
- **Weights were elicited with linear constraints and DSS**
- **Absolute values $\boldsymbol{V_1, \ldots, V_5}$ fixed by assessing tradeoffs between criteria pairs**
- **Optimal portfolio solved with the enumeration algorithm and also with the approximate MILP model for comparison**
  - Approximate MILP was faster

References

1. Liesiö, J., 2014: Measurable Multiattribute Value Functions for Portfolio Decision Analysis, Decision Analysis 11/1, s. 1-20.
2. Golabi, K., Kirkwood, C. W., Sicherman, A., 1981: Selecting a Portfolio of Solar Energy Projects Using Multiattribute Preference Theory, Management Science 27/2, s. 174-189.
3. Bertsimas D, Tsitsiklis JN (1997) Introduction to Linear Optimization, Athena Scientific Series in Optimization and Neural Computation, Vol. 6 (Athena Scientific, Belmont, MA).

Homework

1. **In what situation should you use the additive-multilinear portfolio value function instead of the additive-linear one?**

2. **Calculate criterion-specific $V_1(x_{J1})$ for portfolios {A}, {B,C}, {A, E}, {A,B,C} with given project values**
   a) With additive-linear $V_1(x_{J1})$ and $w_1 = 0.1$
   b) With additive-multilinear $V_1(x_{J1})$ and $w_1(1) = 0.1, w_1(2) = 0.5, w_1(3) = 0.55$
   c) What differences do you see?

Project	$v_1(x_{j1})$	…
A	0.3	…
B	0.8	…
C	0.5	…
D	1	…
E	1	…

DL 22.10. 09:00
Send your answer to suvi.laine@aalto.fi