Aalto University
School of Science

Decision making under uncertainty: modeling alternatives, utility, utility functions, elicitation, incomplete information

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Recap

Additive Value Functions

$$V(x) = \sum_{i=1}^{n} w_i v_i(x_i)$$

Weights w_i and value functions v_i obtained with different elication methods

↓

Compare best values to get best choice

Additive Linear Portfolio Value Function

$$V(X) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_i\, v_i(x_{ji})$$

↓

Formulate an ILP max/min problem with some constraints

$$\max_{z_j\in\{0,1\}} \sum_{i=1}^{n}\sum_{j=1}^{m} w_i\, v_i(x_{ji}) z_j$$

$$s.t \sum_{j=1}^{m} c_j z_j \le C$$

Preference programming

$$w \in \{w | w_i \ge 0, \sum w_i = 1\}$$

$$w_- \le w_i \le w_+$$

Unclear statements by DM regarding weights or values

↓

Preference programming and concepts of dominance

$$\min_{w\in W}\left[V(x^k) - V(x_l)\right] \ge 0$$

$$\max_{w\in W}\left[V(x^k) - V(x_l)\right] > 0$$

Robust Portfolio Modelling

$$\max_{z_j\in\{0,1\}} \sum_{i=1}^{n}\sum_{j=1}^{m} w_i\, v_i(x_{ji}) z_j$$

$$s.t \sum_{j=1}^{m} c_j z_j \le C$$

$$w \in S_w = \{w | Aw \le b\}$$

$$v \in S_v = \{v | v_i \in [\underline{v_i}, \overline{v_i}]\}$$

↓

Methods for calculating the ”robust” projects (Core index)

Recap

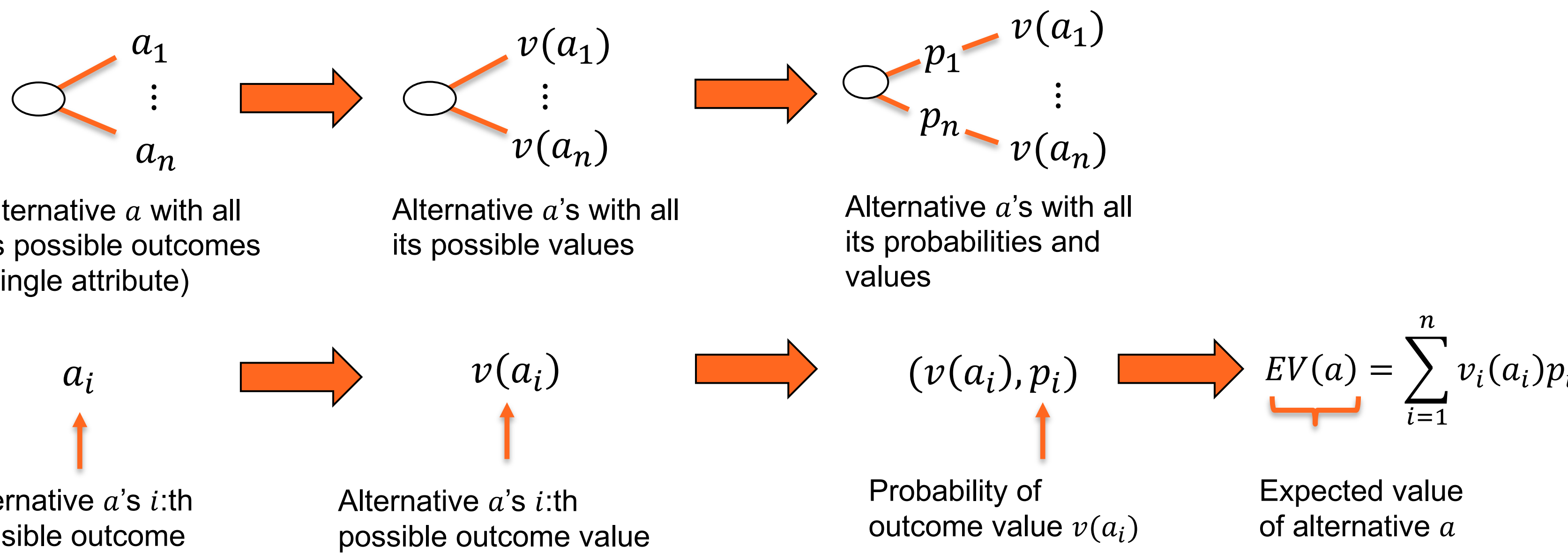
- **So far all methods have assumed:**
  - Outcomes are certain
  - No uncertainty if the projects or alternatives are "successful"
  - No discussion about probabilities
  - Assumed that the DM is risk neutral

- **We have ignored the DM's risk preference and that outcomes are not certain. There is usually risk involved.**

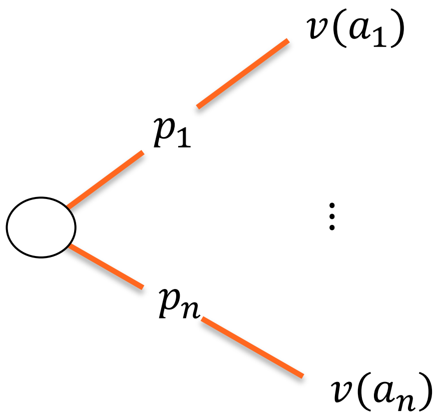
Introduction

Naïve approach…

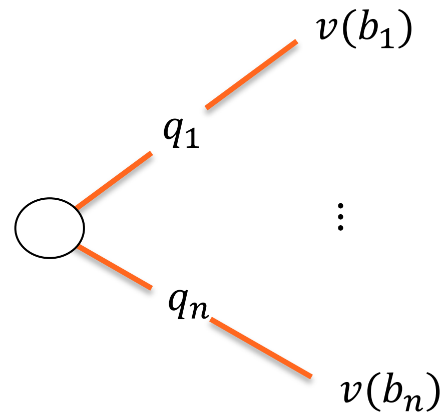
What if we expand on the value functions to include probabilities?



Introduction



Vs.

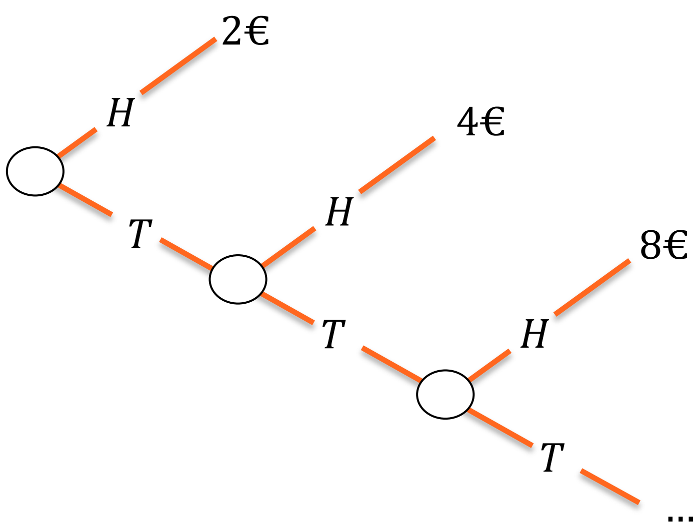


$$EV(a) = \sum_{i=1}^{n} v(a_i)p_i \qquad EV(b) = \sum_{i=1}^{n} v(b_i)q_i$$

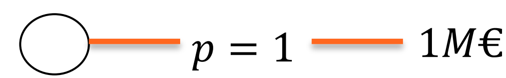
Rational decision maker would choose alternative a over b if $EV(a) > EV(b)$

Introduction

Counter example (“St. Petersburg Lottery”)



Vs.



$EV = \infty$

$EV = 1M€$

Introduction

- **The counter example showed that it does not reflect intutional decision behavior**
  → introduces paradoxes

- **Theory needs to be “expanded” properly**
  → <u>expected utility theory</u>

- **Expected utility theory deals with <u>lotteries</u> and asks the DM to compare these lotteries to map the DM’s risk preferences.**

Expected Utility Theory

- **Introduce the utility function $u(x)$ which represents the DM's attitude towards risk and preference**

- **The utility function $u(x)$ assigns a real number to each consequence x**
  $u(x)$ is normally scaled to a value between 0 and 1.
  $u(x)$ is unique up to a positive affine transformation.

- **As a summary:**

decision under certainty → value function $v(x)$

decision under risk (uncertainty) → utility function $u(x)$

Expected Utility Theory

- **If the DM's preference for risky (uncertain) alternatives fullfills the following criteria:**

1. Complete
2. Transitive
3. Continuity
4. Independence

$\left.\begin{matrix} 1. \\ 2. \\ 3. \\ 4. \end{matrix}\right\}$ **axioms**

Then there exists a utility function $u(x)$, which expected value represents the DM's preferences.

Expected Utility Theory

1. **Completeness**
   $a > b$ or $b > a$ or $b \sim a$

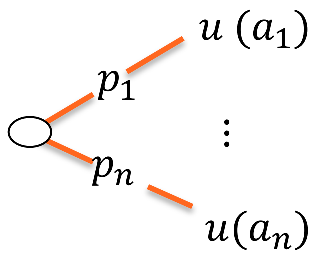
2. **Transitive**
   if $a > b$ and $b > c$ then $a > c$

3. **Continuity**
   if $a > b > c$ then there exists probability p such that
   $b \sim p * a + (1 - p) * c$

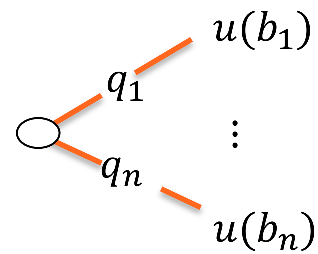
4. **Independence**
   if $a > b$ then for all c and probabilities p there exists
   $p * a + (1 - p) * \mathrm{c} \geq p * b + (1 - p) * c$

Expected Utility Theory

- **The lotteries describes outcomes with certain probabilities**



Vs.



- **Rational DM chooses $\boldsymbol{a}$ over $\boldsymbol{b}$ if $\boldsymbol{EU(a) > EU(b)}$:**

$$\sum_{i=1}^{n} p_i u\,(a_i) > \sum_{i=1}^{n} q_i u(b_i)$$

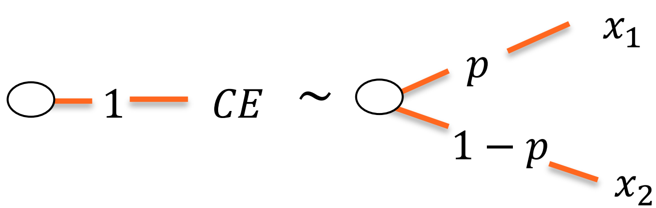
Expected Utility Theory

- **Certainty equivalence (CE):**

$$u\big(CE(a)\big) = EU(a)$$

Expected Utility Theory

- **How to find the utility function?**
  1. Basic reference lotteries



1. 2. 3.

$$u(CE) = p * u(x_1) + (1-p) * u(x_2)$$

4.

4 unknowns...
→ fix 3 and solve for 4th

  2. Bisection methods
  3. Trade-off methods
     ⋮

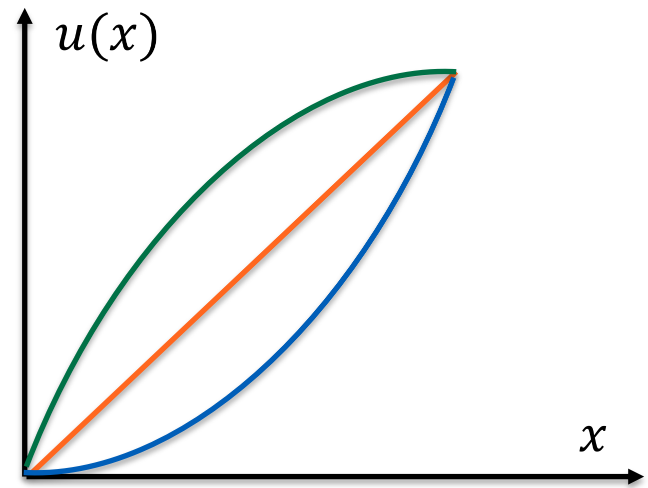
Expected Utility Theory

- **Utility functions curvature represents risk behaviour**

linear $u(x)$ → risk neutral

convex $u(x)$ → risk prone

concave $u(x)$ → risk averse



- **Explaining and proving this is left as an exercise (= homework)!**

Modelling for decision under risk and incomplete information

- **What if the DM cannot state his preferences with lotteries?**
  1. Uncertainty in the probabilities p_i
  2. Uncertainty about the DM's utility function $u(x)$

- **We can form a set of candidate probabilities and utility functions**
  1. $p \in P(I) = \{p_i \mid \underline{p_i} \leq p_i \leq \overline{p_i}\}$
  2. $u \in U(I) = \{u \mid \underline{u}(a_i) \leq u(a_i) \leq \overline{u}(a_i)\}$

Modelling for decision under risk and incomplete information

Uncertain probabilities:

Expand on the idea of preference programming:

$$a \geq b \iff \sum_{i=1}^{n} p_i u(a_i) \geq \sum_{i=1}^{n} p_i u(b_i) \quad \forall\, p \in P(I)$$

Construct a linear optimization problem similar as in preference programming:

$$\max_{p_i} or \min \sum_{i=1}^{n} p_i \big(u(a_i) - u(b_i)\big)$$

$$\underline{p_i} \leq p_i \leq \overline{p_i}$$

$$0 \leq p_i \leq p_j$$

$$\sum_{i=1}^{n} p_i = 1$$

Modelling for decision under risk and incomplete information

Uncertain probabilities:

- Target is to minimize or maximize the difference between the expected values

$$\min_{p_i} \sum_{i=1}^{n} p_i\big(u(a_i) - u(b_i)\big) \qquad \max_{p_i} \sum_{i=1}^{n} p_i\big(u(a_i) - u(b_i)\big)$$

- If $\min\,[EU(a) - EU(b)] \geq 0$ and $\max\,[EU(a) - EU(b)] \geq 0$
  → a preferred over b

- If $\min\,[EU(a) - EU(b)] \leq 0$ and $\max\,[EU(a) - EU(b)] \leq 0$
  → b preferred over a

- If $\min\,[EU(a) - EU(b)] \leq 0$ and $\max\,[EU(a) - EU(b)] \geq 0$ (and vice versa)
  → no preference statement

Modelling for decision under risk and incomplete information

Uncertain utility functions:

Expand on the idea of preference programming:

$$a \geq b \;\Leftrightarrow \sum_{i=1}^{n} p_i u(a_i) \geq \sum_{i=1}^{n} p_i u(b_i) \quad \forall\, u \in U(I)$$

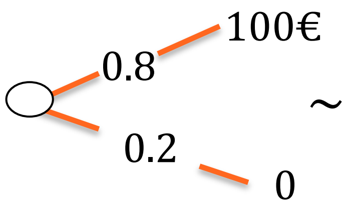
Similarly construct an optimization problem

$$\max_{u(a_i),u(b_i)} or \min \sum_{i=1}^{n} p_i\big(u(a_i) - u(b_i)\big)$$
$$\underline{u}(a_i) \leq u(a_i) \leq \overline{u}(a_i)$$
$$\underline{u}(b_i) \leq u(b_i) \leq \overline{u}(b_i)$$

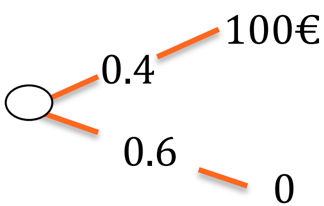
Modelling for decision under risk and incomplete information

- **Uncertain utility functions:**

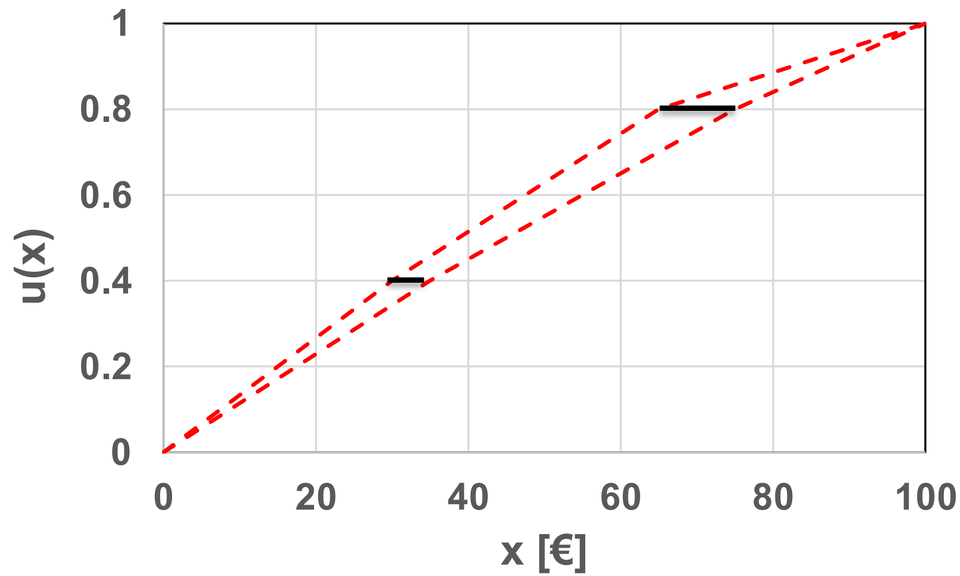
  Example:



$\sim CE \in [65€, 75€]$



$\sim CE \in [30€, 35€]$



Modelling for decision under risk and incomplete information

Uncertain utility functions:

- Target is to minimize or maximize the difference between the expected values

$$\min_u \sum_{i=1}^{n} p_i\big(u(a_i) - u(b_i)\big) \qquad \max_u \sum_{i=1}^{n} p_i\big(u(a_i) - u(b_i)\big)$$

- If $\min\,[EU(a) - EU(b)] \geq 0$ and $\max\,[EU(a) - EU(b)] \geq 0$
  → a preferred over b

- If $\min\,[EU(a) - EU(b)] \leq 0$ and $\max\,[EU(a) - EU(b)] \leq 0$
  → b preferred over a

- If $\min\,[EU(a) - EU(b)] \leq 0$ and $\max\,[EU(a) - EU(b)] \geq 0$ (and vice versa)
  → no preference statement

Modelling for decision under risk and incomplete information

- **However…**

  When maximizing and having equal consequences, i.e. then

$$a_i = b_i$$

then at the optimum we would have

$$\overline{u}(a_i) \text{ and } \underline{u}(b_i)$$

Upper bound

Lower bound

- **Solution is "artificially" maximized**

Modelling for decision under risk and incomplete information

- **Better procedure is needed:**
  For any given alternatives a and b a procedure that determines which is preferred for all permissible utility functions $u(x)$.

- **Concept of Stochastic Dominance**
  1. First degree stochastic dominance
  2. Second degree stochastic dominance

Modelling for decision under risk and incomplete information

First degree stochastic dominance (FSD)

Work with cumulative probability distributions for alternatives.

For all consequences of alternatives (lotteries) a and b, construct a cumulative probability function $P_a(x)$ and $P_b(x)$. If

$$P_a(x) \leq P_b(x) \ \ \forall x$$

$$\forall u \in U_0 = \{u \ is \ strictly \ increasing\}$$

then a dominates b in the sense of first degree stochastic dominance for all, thus

$$a \geq_{FSD} b$$

If an alternative is strictly FSD dominated, then a DM who preferes more to less should not chose the dominated alternative

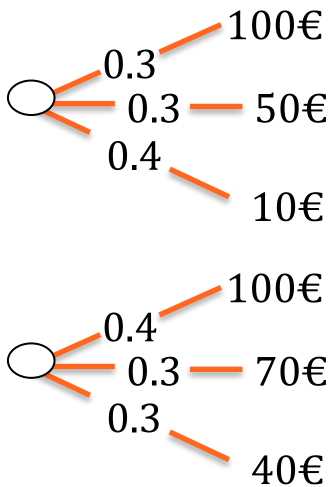
Modelling for decision under risk and incomplete information

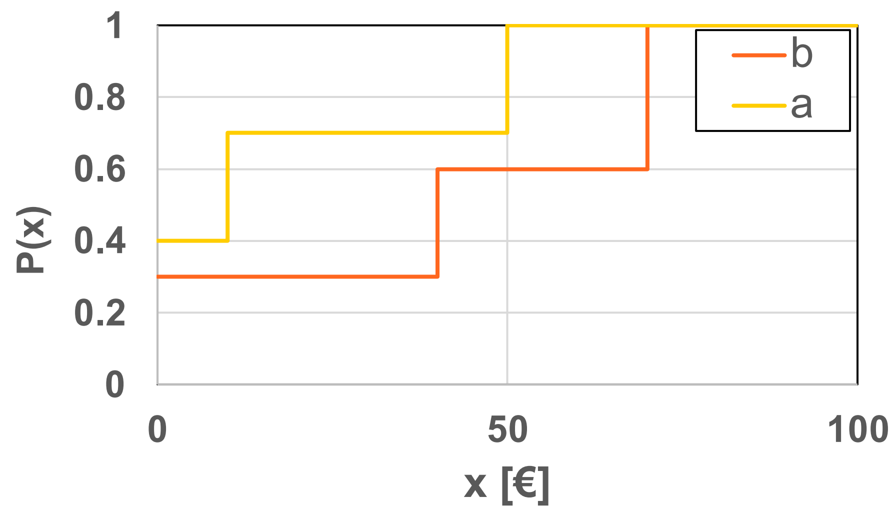
First degree stochastic dominance

Example:

Lottery a:

Lottery b:





$$P_a(x) \geq P_b(x)\ \forall\ x \implies b \geq_{FSD} a$$

Modelling for decision under risk and incomplete information

Second degree stochastic dominance (SSD)

Work with cumulative probability distributions for alternatives.

For all consequences of alternatives (lotteries) a and b, construct a cumulative probability function $P_a(x)$ and $P_b(x)$. If

$$\int_{-\infty}^{x} [P_a(t) - P_b(t)]dt \ \leq 0 \ \ \forall x \ ,$$

$$\forall u \in U_0 = \{u \ is \ concave\}$$

then a dominates b in the sense of second degree stochastic dominance, thus

$$a \geq_{SSD} b$$

A DM who prefers more to less and is risk averse or risk neutral should not choose a SSD dominated alternative.

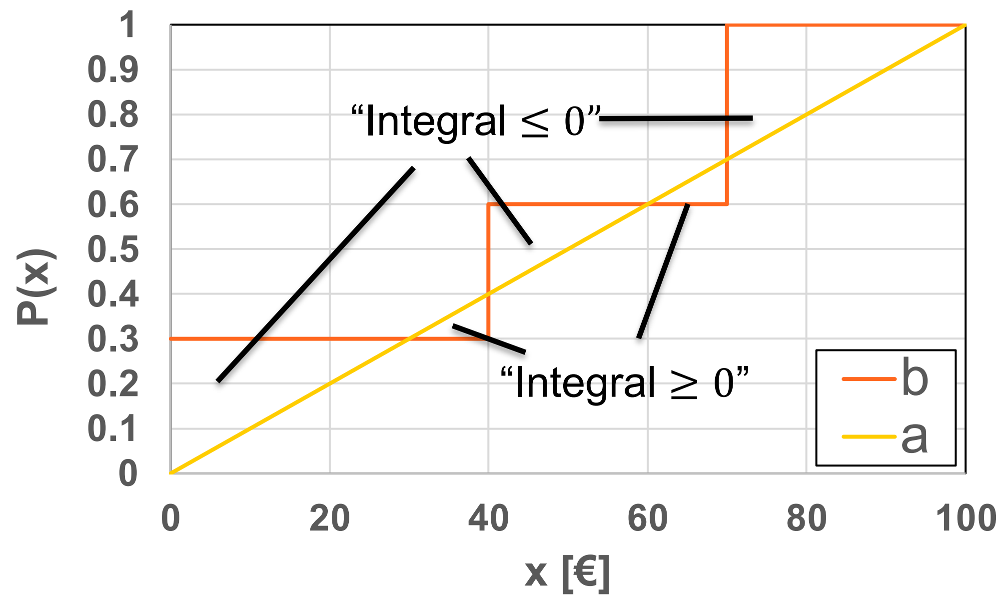
Modelling for decision under risk and incomplete information

Second degree stochastic dominance

Example:

$$\int_{-\infty}^{x} [P_a(t) - P_b(t)]dt \le 0 \ \forall x$$

$$\Rightarrow a \ge_{SSD} b$$



Modelling for decision under risk and incomplete information

- **Properties of stochastic dominance:**

  - If a DM prefers more to less and $a \geq_{FSD} b$ then the DM should <u>not</u> choose alternative b, since $EU[a] \geq EU[b] \; \forall \; u \in U_0$ where U_0 is the set of all strictly increasing function.

  - If a $a \geq_{SSD} b$ and the DM is risk avers or neutral, and prefers more to less, then the DM should not choose alternative b, since $EU[a] \geq EU[b] \; \forall \; u \in U_0$ where U_0 is the set of all concave functions.

  - FSD implies automatically SSD.

  - FSD and SSD are transitive properties.

Summary

- **We successfully expanded decision making under certainty to uncertainty: $v(x) \rightarrow u(x)$**

- **We discussed methods on how to find $u(x)$ (needed axioms, concept of lotteries, CE etc.)**

- **We discussed modelling under uncertainty**

- **We introduced new tools for determening dominance of alternatives (FSD, SSD)**

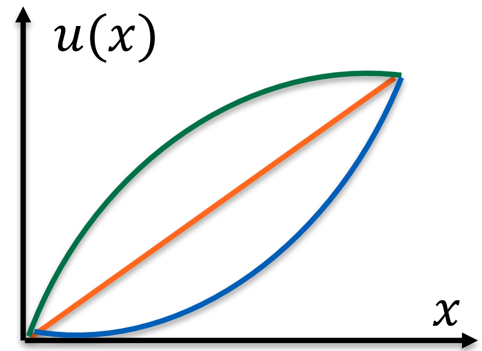
References

- Eisenführ, F., Weber, M., Langer, T., 2010: *Rational Decision Making*, Springer-Verlag Berlin Heidelberg.
- Punkka, Liesiö, Salo, Vilkkumaa , 2020, *Decision Making and Problem Solving - Lecture 2*, lecture slides, Decision Making and problem solving MS-E2134, Aalto University, delivered Spring 2020.
- Punkka, Liesiö, Salo, Vilkkumaa , 2020, *Decision Making and Problem Solving - Lecture 3*, lecture slides, Decision Making and problem solving MS-E2134, Aalto University, delivered Spring 2020.

Homework

Prove that a) concave and b) convex utilitiy functions state the DM's risk prone or risk averse attitude.

Hint: Check Eisenführ et al. page 252-254.
If you don't have access to the book, please send an email to me and I will send you the relevant pages.



Send the answers to oliver.lundqvist@aalto.fi (DL 5.11.2021 9.00)
Try to use the subject "OR hw10" when submitting your homework.