Model Solution for Homework 10

Prove that a) concave and b) convex utility functions state the DM's risk prone or risk averse attitude.

Let's show this with an example. Assume that a decision maker (DM) has two possible outcomes of a and b of a lottery. The outcomes are measured on a numerical scale (i.e. they are real numbers) and $b > a$. Both outcomes a and b have a utility of $u(a)$ and $u(b)$ respectively. The two outcomes can be shown in Figure 1.

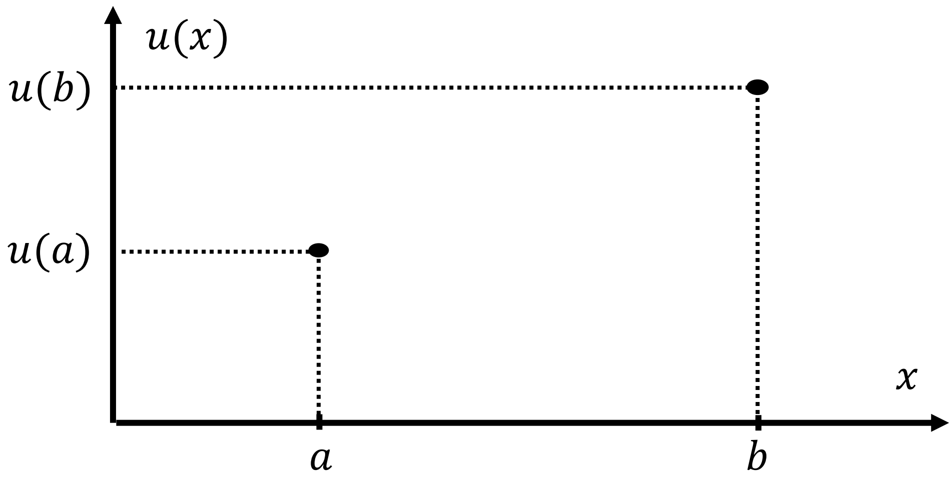

Figure 1. Utilities of alternatives a and b.

Assume that outcome a will happen with a probability of p and b will happen with a probability of $1 - p$. Thus, the expected value (EV) of the lottery will be

$$EV = pa + (1 - p)b \quad (1)$$

which is a convex combination of the two possible outcomes of a and b. The expected value and the utility of the expected value can be drawn in the graph by connecting a line between a and b and setting the expected value point on that line such that the point divides the line into a p and $1 - p$ long lines. The expected value is shown in Figure 2

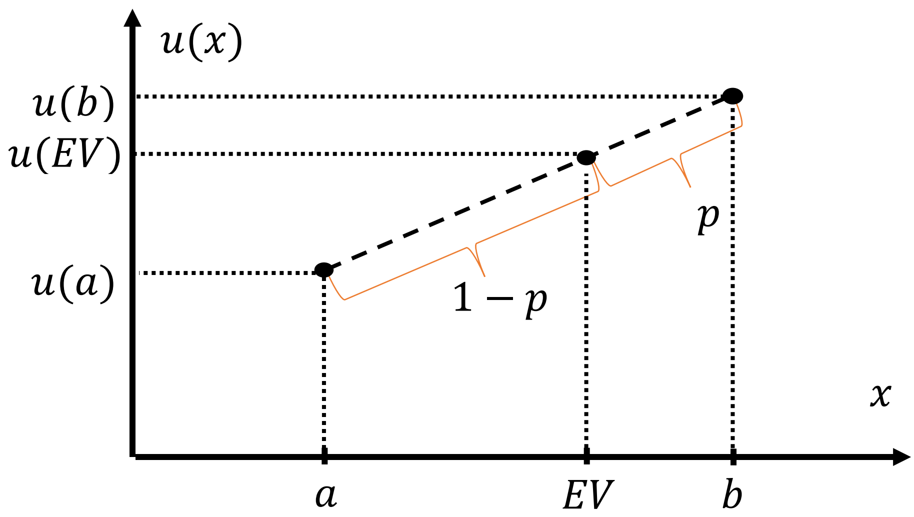

Figure 2. Expected value.

Now, let's assume that our utility function is concave. By the definition of a concave function, we have the following statement for the utility function:

$$u(pa + (1-p)b) > pu(a) + (1-p)u(b) \qquad (2)$$

From equation (2) we see that the left hand side is actually the utility at the expected value and the right hand side the expected utility, hence

$$u(EV) > EU \qquad (3)$$

We also know that a DM should be indifferent between participating to a lottery and the certainty equivalent (CE) amount. Thus, the expected utility of this lottery is the same as the utility of the certainty equivalent and we can state

$$EU = u(CE) \qquad (4)$$

If we combine the information from equation (3) and (4) into a similar graph as in Figures 1 and 2, we can see that the certainty equivalent CE is lower than the expected value EV. The graph is shown in Figure 3. The DM is willing to choose a "lower amount of x" in order to avoid the "gamble". Thus, a DM with a concave utility function is risk averse.

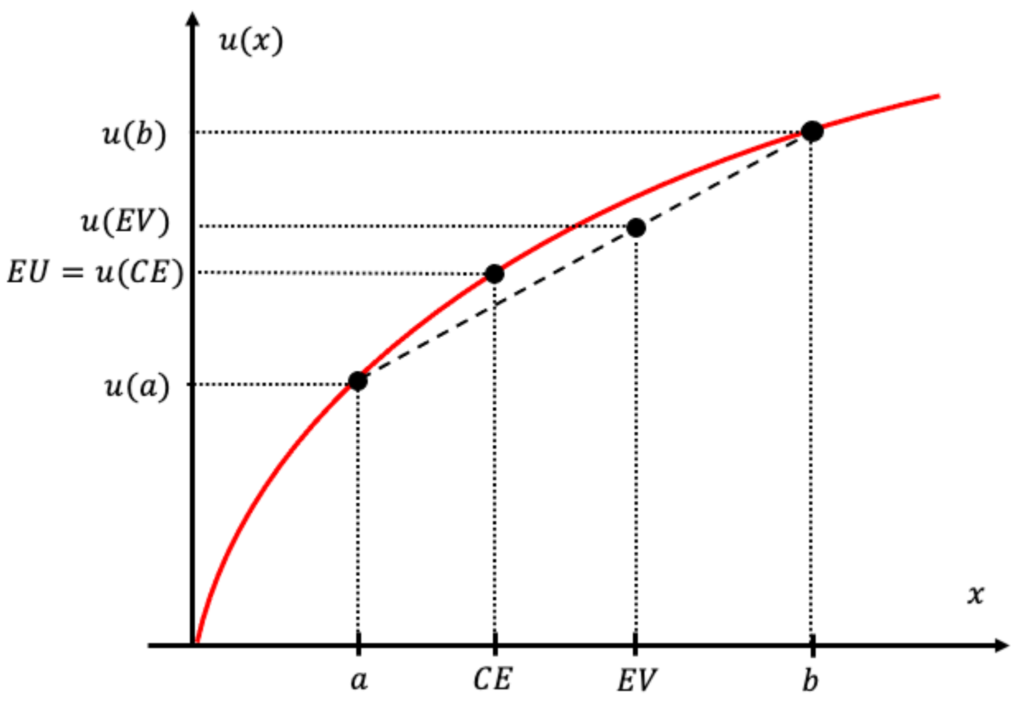

Figure 3. Concave utility function and the expected values and utilities.

Similarly for a convex utility function, the inequality (2) has a reversed the inequality sign, hence

$$u(pa + (1-p)b) < pu(a) + (1-p)u(b) \qquad (5)$$

Which leads to

$$u(EV) < EU = CE \tag{6}$$

Therefore, the DM with a convex utility function has certainty equivalent CE higher than the expected value EV of the lottery, which results in risk seeking or risk prone attitude.