Aalto University
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(Nonadditive) multiattribute utility functions for Portfolio Decision Analysis

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Previously

Nonadditive portfolio value functions:

- Depends on the number of included projects
- Attribute specific value function $v_i(x_{ji})$
- Strictly increasing weighting function $w_i(1), \dots, w_i(m), w_i(0) = 0$

=> Additive multilinear value function:

$$V_i(x_{Ji}) = \sum_{J' \subseteq J} w_i(|J'|) \prod_{j \in J'} v_i(x_{ji}) \prod_{j \notin J'} (1 - v_i(x_{ji}))$$

Liesiö, 2014

Decision making under uncertainty:

- The project outcome is not certain
- There were probabilities between outcomes
- Replace value function $v(x)$ with utility function $u(x)$
- Expected utility theory

Today we will combine these

Notation

Variable	Description
m	Number of projects
n	Number of attributes in projects
x_j	Deterministic project, $j \in \{1, m\}$
$\tilde{x}_j$	Nondeterministic project, $j \in \{1, m\}$
$x = (x_1, \dots, x_m)$	Set of projects, portfolio
$y = (y_1, \dots, y_n)$	Outcome of project
$y^0 = (y_1^0, \dots, y_n^1)$	The least preferred outcome of project
$y^* = (y_1^*, \dots, y_n^*)$	The most preferred outcome of project

Example

- Healthcare resource allocation problem
- Each project is uncertain and has certain cost
- Attributes:
  - Health benefits
  - Health inequality reduction

Each project has uncertain outcome!

Table 1. Expected Multiattribute Utilities (Airoldi et al. 2011) and Costs of the Intervention Projects

Project	j	$\mathbb{E}[u(\tilde{x}_j^F)]$	c_j (k£)	$\mathbb{E}[u(\tilde{x}_j^F)]/c_j$
Pneumonia	1	0.7	75	0.1579
Dementia services	2	0.31	50	0.1036
TIA and secondary prevention	3	0.32	130	0.0415
Prison MH	4	0.27	150	0.0301
Obesity training	5	0.1	60	0.0288
Workforce development	6	0.16	100	0.0278
Psych therapies	7	0.18	120	0.0254
Early detection and diagnostics	8	0.34	300	0.0191
CAMHS school	9	0.16	160	0.0172
Prevention	10	0.62	650	0.0161
CAMHS 1:1	11	0.07	80	0.0158
Cardiac rehab	12	0.08	100	0.0129
Alcohol misuse svc	13	0.22	300	0.0126
Social inclusion	14	0.22	300	0.0125
Palliative and EOL	15	0.54	760	0.0119
Obesity 1:1	16	0.07	140	0.0087
Primary prevention	17	0.27	600	0.0077
Access to dental	18	0.19	480	0.0068
Active treatment	19	0.02	50	0.0062
Stroke emergency	20	0.2	600	0.0056
CHD acute	21	0.05	300	0.0026

Note. The projects are listed in a decreasing order of utility-to-cost ratios $\mathbb{E}[u(\tilde{x}_j^F)]/c_j$.

Liesiö and Vilkkumaa, 2021

Airoldi et al., 2011

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Assumptions 1 & 2

Assumption 1: Preferences are independent from project indexing:

$$(x_1, x_2, \dots, x_{j-1}, x_j, x_{j+1}, \dots) \sim (x_j, x_2, \dots, x_{j-1}, x_1, x_{j+1}, \dots)$$

Assumption 2: Preferences for uncertain project outcomes do not depend on the deterministic outcomes of the other projects:

$$(\tilde{x}_1, x_2, x_3, \dots) \succcurlyeq (\tilde{x}_1', x_2, x_3, \dots) \Rightarrow (\tilde{x}_1, x_2', x_3', \dots) \succcurlyeq (\tilde{x}_1', x_2', x_3', \dots)$$

Multilinear Utility Function

Theorem 2 (Liesiö and Vilkkumaa, 2021): The assumptions 1 & 2 are satisfied if and only if the portfolio utility function $U : X \rightarrow \mathbb{R}$ is multilinear:

$$U(x_1, \ldots, x_m) = \sum_{J \subseteq \{1,\ldots,m\}} \lambda(|J|) \prod_{j \in J} u(x_j) \prod_{j \notin J} (1 - u(x_j))$$

$\lambda(k)$	Weight function
J	Subset of chosen portfolio
$\lvert J \rvert$	Number of included projects in the subset
$u(x_j)$	Utility function

$$u(y) = U(y, y^0, \ldots, y^0)$$

$$\lambda(0) = U(y^0, \ldots, y^0) = 0$$

$$\lambda(1) = U(y^*, y^0, \ldots, y^0) = 1$$

$$\lambda(k) = U\left(\underbrace{y^*, \ldots, y^*}_{k}, y^0, \ldots, y^0\right)$$

Numerical example

Project	$u(x_j)$
x_1	0.2
x_2	0.3
x_3	0.5

$\lvert J \rvert$	$\lambda(k)$
1	1.00
2	1.25
3	1.50

$$U(x_1, \ldots, x_m) = \sum_{J \subseteq \{1,\ldots,m\}} \lambda(|J|) \prod_{j \in J} u(x_j) \prod_{j \notin J} (1 - u(x_j))$$

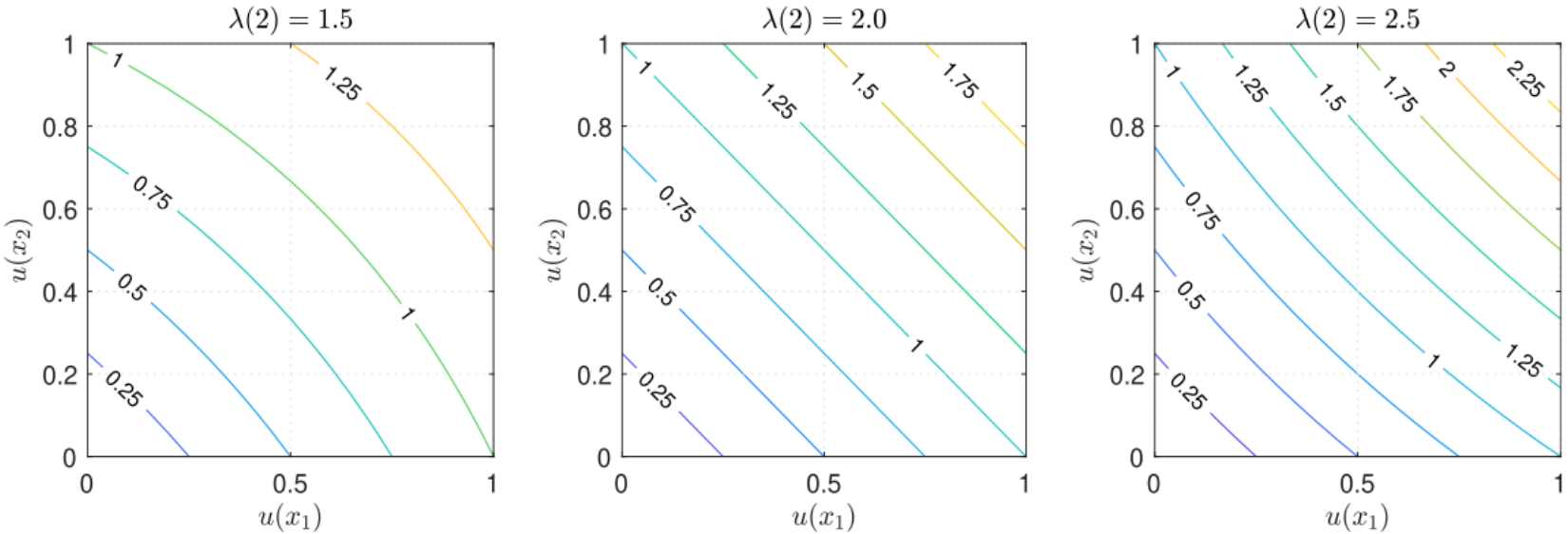
$$U(x_1) = 1 * 0.2 = 0.2$$

$$\begin{aligned} U(x_1, x_2) &= 1 * 0.2 * (1 - 0.3) + 1 * 0.3 * (1 - 0.2) + 1.25 * 0.3 * 0.2 \\ &= 0.455 \end{aligned}$$

$$\begin{aligned} U(x_1, x_2, x_3) &= 1 * 0.2 * 0.7 * 0.5 + 1 * 0.3 * 0.8 * 0.5 + 1 * 0.5 * 0.7 * 0.8 \\ &\quad + 1.25 * 0.2 * 0.3 * 0.5 + 1.25 * 0.2 * 0.5 * 0.7 \\ &\quad + 1.25 * 0.3 * 0.5 * 0.8 + 1.5 * 0.2 * 0.3 * 0.5 \\ &= 0.79 \end{aligned}$$

Different λ values

The value of multilinear utility function changes with respect to λ



Liesiö and Vilkkumaa, 2021

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Preference elicitation: Approach 1

DM will be asked to compare two portfolios with different types, deterministic and uncertain:

$$x = \left(\underbrace{y^*, \dots, y^*}_{k}, y^0, \dots, y^0\right)$$

$$\tilde{x} = \begin{cases} \left(\underbrace{y^*, \dots, y^*}_{k-1}, y^0, \dots, y^0\right), \text{with probability } 1-p \\ \left(\underbrace{y^*, \dots, y^*}_{k+1}, y^0, \dots, y^0\right), \text{with probability } p \end{cases}$$

$$\text{for } k \in \{1, \dots, m-1\}$$

What is the probability p that DM is indifferent in between these portfolios?

Preference elicitation: Approach 1

We can evaluate expected utilities $\mathbb{E}[u(x)]$ and $\mathbb{E}[u(\tilde{x})]$ for $k \in \{1, \dots, m-1\}$.

$$\begin{aligned}
\mathbb{E}[u(x)] &= \mathbb{E}[u(\tilde{x})] \\
U\left(\underbrace{y^*, \dots, y^*}_{k}, y^0, \dots, y^0\right) &= (1-p)U\left(\underbrace{y^*, \dots, y^*}_{k-1}, y^0, \dots, y^0\right) + pU\left(\underbrace{y^*, \dots, y^*}_{k+1}, y^0, \dots, y^0\right) \\
\Leftrightarrow \lambda(k) &= (1-p)\lambda(k-1) + p\lambda(k+1) \\
\Leftrightarrow \lambda(k+1) &= \left(\frac{1}{p} - 1\right)(\lambda(k) - \lambda(k-1)) + \lambda(k)
\end{aligned}$$

We know $\lambda(0) = 0$ and $\lambda(1) = 1$.

$\Longrightarrow$ We have $m-1$ variables and equations

Preference elicitation: Approach 2

- Bisection-type preference elicitation
- Basic idea: The DM is asked to define the probability p such that

$$\dot{x}^k \sim p\dot{x}^{\overline{k}} + (1-p)\dot{x}^{\underline{k}}, \qquad k = \lfloor (\overline{k} + \underline{k})/2 \rfloor, \qquad \overline{k}, \underline{k} \in \{0, \dots, m\}$$

$$\dot{x}^k = (\underbrace{y^+, \dots, y^+}_{k}, y^-, \dots, y^-)$$

$$\dot{x}^{\overline{k}} = (\underbrace{y^+, \dots, y^+}_{\overline{k}}, y^-, \dots, y^-)$$

$$\dot{x}^{\underline{k}} = (\underbrace{y^+, \dots, y^+}_{\underline{k}}, y^-, \dots, y^-)$$

$$y^0 \prec y^- \preccurlyeq y^+ \prec y^*$$

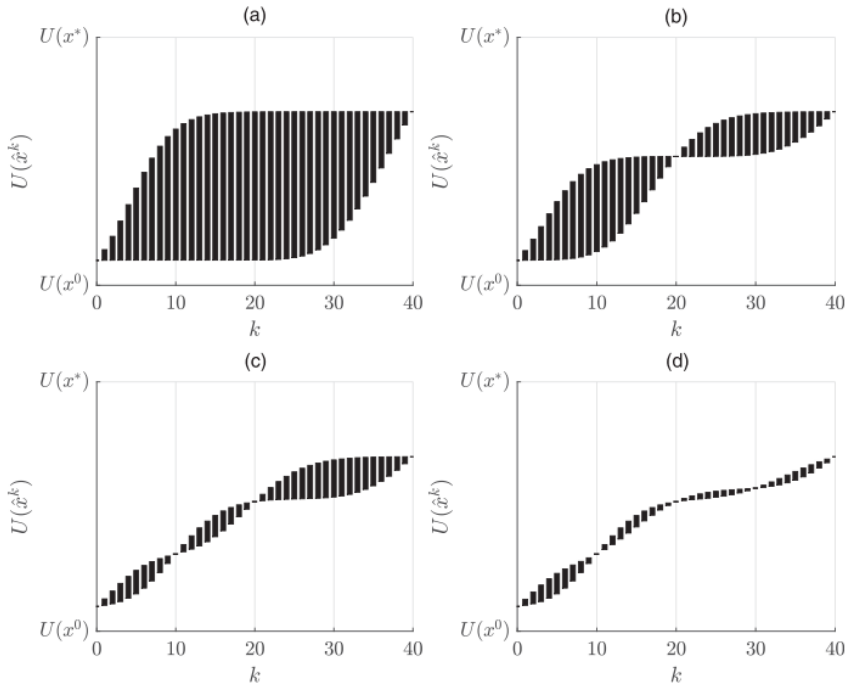
Preference elicitation: Approach 2

$$\dot{x}^k \sim p\dot{x}^{\overline{k}} + (1-p)\dot{x}^{\underline{k}}, \qquad k = \lfloor(\overline{k} + \underline{k})/2\rfloor$$

First compare $\dot{x}^0 = (y^-, \dots, y^-) \sim px^* + (1-p)x^0$
Second compare $\dot{x}^m = (y^+, \dots, y^+) \sim px^* + (1-p)x^0$

- Third $\overline{k} = m,\ \underline{k} = 0$
- Fourth $\overline{k} = \lfloor m/2 \rfloor,\ \underline{k} = 0$
- Fifth $\overline{k} = m,\ \underline{k} = \lfloor m/2 \rfloor$, and so on…

Each statement gives $\lambda(0), \dots, \lambda(m)$



Liesiö and Vilkkumaa, 2021

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Assumption 3

Assumption 3: Portfolios x and $\tilde{x}$ are equally preferred for any $k = \{2, \dots, m\}$:

$$x = \left(\underbrace{y^*, \dots, y^*}_{k}, y^0, \dots, y^0\right)$$

$$\tilde{x} = \begin{cases} \left(\underbrace{y^*, \dots, y^*}_{k-1}, y^0, \dots, y^0\right), \text{with probability } 0.5 \\ \left(\underbrace{y^*, \dots, y^*}_{k+1}, y^0, \dots, y^0\right), \text{with probability } 0.5 \end{cases}$$

Theorem 2 (Liesiö and Vilkkumaa, 2021): Assumptions 1, 2 and 3 hold if and only if portfolio utility function is **additive**:

$$U(x) = \sum_{j=1}^{m} u(x_j)$$

With assumptions 1-3 we have additive independence assumed

Assumption 4

Assumption 4: There exists $p^* \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ such that portfolios x and $\tilde{x}$ are equally preferred for any $k = \{2, \dots, m\}$:

$$x = \left(\underbrace{y^*, \dots, y^*}_{k}, y^0, \dots, y^0\right)$$

$$\tilde{x} = \begin{cases} \left(\underbrace{y^*, \dots, y^*}_{k-1}, y^0, \dots, y^0\right), \text{with probability } p^* \\ \left(\underbrace{y^*, \dots, y^*}_{k+1}, y^0, \dots, y^0\right), \text{with probability } 1 - p^* \end{cases}$$

Theorem 3 (Liesiö and Vilkkumaa, 2021): Assumptions 1, 2 and 4 hold if and only if portfolio utility function is **multiplicative**:

$$U(x) = \frac{1}{\theta} \prod_{j=1}^{m} \left(1 + \theta u(x_j)\right) - \frac{1}{\theta}$$

Where $\theta = \frac{1}{p^*} - 2, \quad \theta \in (-1, 0) \cup (0, \infty)$

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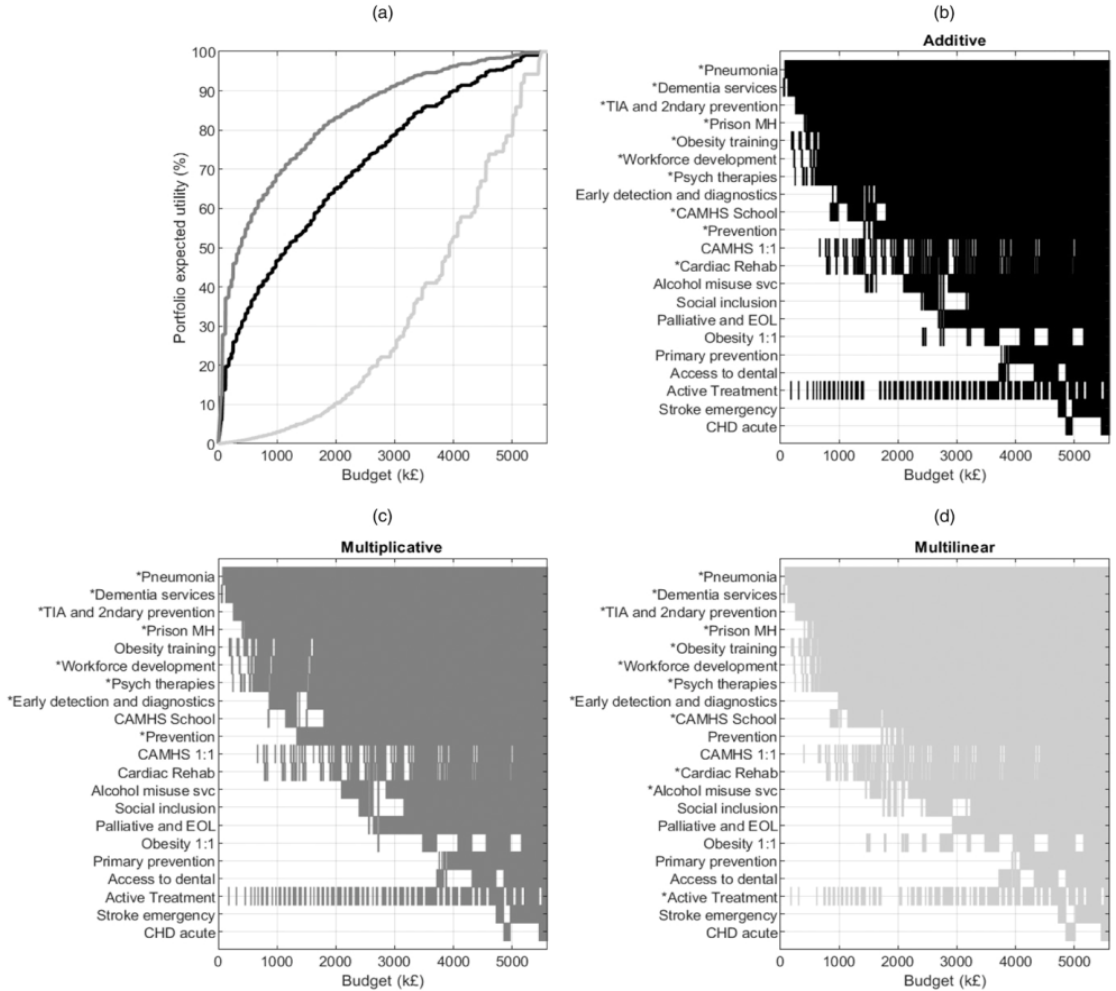
Example

- Healthcare resource allocation solved with three portfolio utility functions
- Budget from 0 k£ to 5505 k£
- How different portfolio utility functions choose included projects?



Notes. The budget levels for which a particular project is included in the optimal portfolio are colored. Projects that are included in the optimal portfolio at budget level £1.6 million are indicated with asterisks. The projects are listed in a decreasing order of utility-to-cost ratios $\mathbb{E}[u(\tilde{x}_j^t)]/c_j$.

Liesiö and Vilkkumaa, 2021

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Summary

- Multiattribute utility functions for uncertain project outcomes
- Multilinear utility function
- Eliciting the weights λ
- Special cases: additive and multiplicative utility functions
- Maximizing the expected utility

References

- Liesiö, J., Vilkkumaa, E., 2021. Nonadditive Multiattribute Utility Functions for Portfolio Decision Analysis, Operations Research (to appear).
- Liesiö, J., 2014: Measurable Multiattribute Value Functions for Portfolio Decision Analysis, Decision Analysis 11/1, s. 1-20.
- Airoldi M, Morton A, Smith J, Bevan G (2011) Healthcare prioritisation at the local level: A socio-technical approach. Working paper, University of Oxford.

Homework

1. Describe your own portfolio decision problem that has uncertain projects (projects, attributes, goal of the portfolio).
2. What are advantages and disadvantages of preference elicitation approach 1?
3. What are advantages and disadvantages of preference elicitation approach 2?

Send the homework to matias.peltoketo(at)aalto.fi by 9am on 10th of November with title “Homework 12”.