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Other DEA-models and cross efficiency concept

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Content

- Cross efficiency concept
- Other DEA-models
  - BCC model
  - Additive model
  - SBM model

Cross efficiency concept
Recap of the CCR DEA-model

- DMU *k* finds weights $\boldsymbol{v}_k$, $\boldsymbol{u}_k$ that maximizes its efficiency θ_k

$$\max_{\boldsymbol{v}_k, \boldsymbol{u}_k} . \quad \theta_k = \frac{\sum_i u_{ki} y_{ki}}{\sum_i v_{ki} x_{ki}}$$

$$\text{s.t.} \quad \frac{\sum_i u_{ki} y_{ji}}{\sum_i v_{ki} x_{ji}} \leq 1 \qquad \forall j \in 1, ..., n$$

$$\boldsymbol{v}_k, \boldsymbol{u}_k \geq 0$$

- Now a ”maverick” DMU can attain high efficiency θ

Cross efficiency concept
Peer-appraisal

- We can also evaluate another DMU *s* by using the optimal weights for DMU *k*
- This concept is called peer-appraisal
- The resulting value is called cross efficiency, E_{ks}

$$E_{ks} = \frac{\sum_i u_{ki} y_{si}}{\sum_i v_{ki} x_{si}}$$

Cross efficiency concept
Matrix of cross efficiencies

- We can gather all cross efficiencies into matrix
- Note that $E_{kk} = \theta_k$
  - *E_{kk} also called simple efficiency*
- Averaged appraisal by peers for *k* is obtained by

$$e_k = 1/(n-1)\sum_{s\neq k} E_{sk}$$

Rating DMU	Rated DMU 1	2	3	4	5	6	Averaged appraisal of peers
1	**E_{11}**	E_{12}	E_{13}	E_{14}	E_{15}	E_{16}	A_1
2	E_{21}	**E_{22}**	E_{23}	E_{24}	E_{25}	E_{26}	A_2
3	E_{31}	E_{32}	**E_{33}**	E_{34}	E_{35}	E_{36}	A_3
4	E_{41}	E_{42}	E_{43}	**E_{44}**	E_{45}	E_{46}	A_4
5	E_{51}	E_{52}	E_{53}	E_{54}	**E_{55}**	E_{56}	A_5
6	E_{61}	E_{62}	E_{63}	E_{64}	E_{65}	**E_{66}**	A_6
Averaged appraisal by peers (peer appraisal)	e_1	e_2	e_3	e_4	e_5	e_6	

Table source: Doyle, J., Green, R., 1994. Efficiency and Cross-efficiency in DEA: Derivations, Meanings and Uses, Journal of the Operational Research Society 45/5, s. 567-578.

Cross efficiency concept

Uses of cross efficiency

$$e_k = 1/(n-1) \sum_{s \neq k} E_{sk}$$

- The averaged peer appraisal, e_k can be used to assess DMU k in more objective way
- A DMU that achieves high peer appraisal is considered as a good overall performer
- A DMU that rates itself high (E_{kk}), but has low peer appraisal (e_k) is considered a "maverick" DMU
- A Maverick index M_k can be used to effectively distinguish niche performers among the DMUs of interest.

$$M_k = (E_{kk} - e_k)/e_k$$

Cross efficiency concept

Uses of cross efficiency

- In CCR DEA there can be multiple sets of weights, $\boldsymbol{v_k}$, $\boldsymbol{u_k}$
  - The LP algorithm can stop at any of these weights
- We can define a secondary goal for DEA to minimize the averaged appraisal of peers for *k*, A_k, to select best optimal weights for *k*
  - This formulation is called aggressive method
  - A benevolent method would maximize A_k

$$A_k = 1/(n-1)\sum_{s \neq k} E_{ks}$$

Cross efficiency concept
Aggressive CCR using cross efficiency

- Step 1: Calculate $E_{jj} = \theta_j$ for every DMU $j \in 1 \ldots n$
- Step 2:

$$\begin{aligned}
\min_{\boldsymbol{v}_k, \boldsymbol{u}_k} . \quad & (n-1)A_k = \sum_{s \neq k} E_{ks} = \sum_{s \neq k} \frac{\sum_i u_{ki} y_{si}}{\sum_i v_{ki} x_{si}} \\
\text{s.t.} \quad & E_{ks} = \frac{\sum_i u_{ki} y_{si}}{\sum_i v_{ki} x_{si}} \leq 1 \qquad \forall s \neq k \\
& \frac{\sum_i u_{ki} y_{ki}}{\sum_i v_{ki} x_{ki}} = E_{kk} \\
& \boldsymbol{v}_k, \boldsymbol{u}_k \geq 0
\end{aligned}$$

- Now the weights $\boldsymbol{v}_k, \boldsymbol{u}_k$ are deterministic and best for k

Cross efficiency concept
Summary

- Cross efficiency as a concept is not used very widely in applications
- However, cross efficiency can provide powerful tools
  - Cross efficiency is a great tool for spotting maverick DMUs
  - Cross efficiency provides more objective measure for efficiency
  - Cross efficiency can be used to set a secondary goal for CCR to obtain deterministic weights
- Cross efficiency does all this while maintaining the “hands off” quality of DEA

Other DEA-models

Other DEA-models
Recap of the CCR DEA-model

- We have the primal (multiplier) and dual (envelopment) form of the model

Primal

$$\begin{aligned} \max_{\boldsymbol{v}_k, \boldsymbol{u}_k} \quad & \boldsymbol{u}\boldsymbol{y}_k \\ \text{s.t.} \quad & \boldsymbol{v}\boldsymbol{x}_k = 1 \\ & \boldsymbol{u}Y \leq \boldsymbol{v}X \\ & \boldsymbol{u}, \boldsymbol{v} \geq 0 \end{aligned}$$

$\Rightarrow$

Dual

$$\begin{aligned} \min_{\theta, \boldsymbol{\lambda}} \quad & \theta \\ \text{s.t.} \quad & \theta \boldsymbol{x}_k \geq X\boldsymbol{\lambda} \\ & Y\boldsymbol{\lambda} \geq \boldsymbol{y}_k \\ & \boldsymbol{\lambda} \geq 0 \end{aligned}$$

Other DEA-models
Motivation for other DEA methods

- CCR assumes *constant* returns-to-scale, that is, if **(x,y)** is CCR-efficient, then **(tx,ty)** is also CCR-efficient
  - This assumption does not hold in all applications
- We modify this assumption to introduce the possibility of *variable* returns-to-scale by adding a constraint to CCR that makes the efficient frontier convex

$$\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geq 0,\ \forall j$$

Other DEA-models
BCC model

- Adding the convexity constraint to CCR we get the BCC model (Banker, Charnes and Cooper)
- On the right is the input-oriented BCC model
- Like CCR, BCC measures radial efficiency θ_B from the origin
- Convex efficient frontier
- Variable returns-to-scale

$$
\begin{aligned}
\min_{\theta_B,\boldsymbol{\lambda}} \quad & \theta_B \\
\text{s.t.} \quad & \theta_B \boldsymbol{x}_k \geq X\boldsymbol{\lambda} \\
& Y\boldsymbol{\lambda} \geq \boldsymbol{y}_k \\
& \boldsymbol{e}\boldsymbol{\lambda} = 1 \\
& \boldsymbol{\lambda} \geq 0
\end{aligned}
$$

Other DEA-models
BCC model, Example

$$X = \begin{pmatrix} 2 & 3 & 5 & 4 \end{pmatrix},$$

$$Y = \begin{pmatrix} 1 & 4 & 6 & 3 \end{pmatrix}$$

- BCC-efficiency for D (4,3)

$$\begin{aligned} \min_{\theta_B, \boldsymbol{\lambda}} \quad & \theta_B \\ \text{s.t.} \quad & 4\theta_B \geq 2\lambda_1 + 3\lambda_2 + 5\lambda_3 + 4\lambda_4 \\ & 1\lambda_1 + 4\lambda_2 + 6\lambda_3 + 3\lambda_4 \geq 3 \\ & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0 \end{aligned}$$

$$\boxed{\begin{aligned} \min_{\theta_B, \boldsymbol{\lambda}} \quad & \theta_B \\ \text{s.t.} \quad & \theta_B \boldsymbol{x}_k \geq X\boldsymbol{\lambda} \\ & Y\boldsymbol{\lambda} \geq \boldsymbol{y}_k \\ & \boldsymbol{e}\boldsymbol{\lambda} = 1 \\ & \boldsymbol{\lambda} \geq 0 \end{aligned}}$$

- Solution is

$$\theta_B^* = 0.67, \lambda_1^* = 0.33, \lambda_2^* = 0.67, \lambda_3^* = \lambda_4^* = 0$$

Other DEA-models
BCC model, Properties

- Due to the stricter convexity constraint, θ_B is higher for D in BCC than in CCR case
  - Generally, the CCR-efficiency does not exceed BCC-efficiency
- **Theorem:** A DMU *k* is BCC-efficient if it satisfies:
  1. $\theta_B^* = 1$
  2. $s^{-*} = 0$ and $s^{+*} = 0$ (obtained by the two-phase method)
- **Theorem:** A DMU that has a minimum input value for any input item, or a maximum output value for any output item, is BCC-efficient

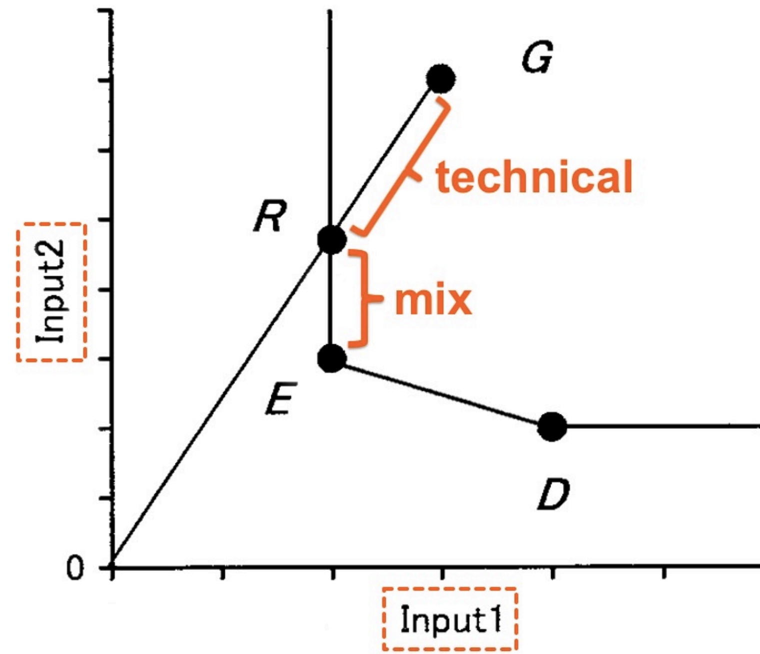
Other DEA-models

From technical efficiency to mix efficiency

- With CCR and BCC models we have measured “technical efficiency” or “radial efficiency”
  - Proportions from origin

$$\theta_B^* = \frac{O\hat{D}}{OD} = \frac{2.67}{4} = 0.67$$

  - Need to define orientation (input or output oriented)
- We move on to measuring mix efficiency directly
  - Proportions are not fixed
  - Input and output side simultaneously



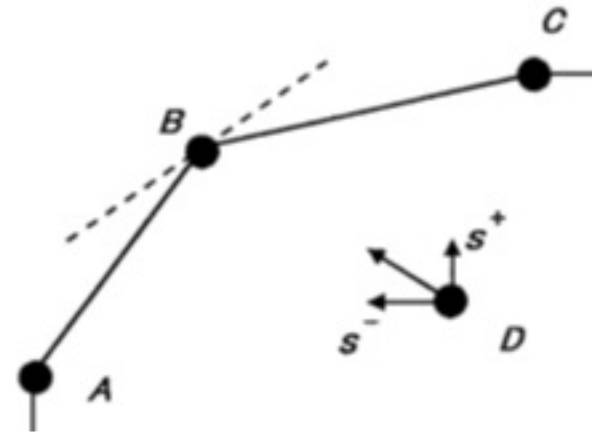
Other DEA-models
Additive model

- The Additive model maximizes input excesses and output shortfalls simultaneously
- The same production possibility set with BCC
- Efficiency *z* measures mix efficiency and is **not** units invariant
- **Theorem:** DMU *k* is ADD-efficient if and only if $\boldsymbol{s}^{-*} = 0$ and $\boldsymbol{s}^{+*} = 0$

$$
\begin{aligned}
\max_{\boldsymbol{\lambda}, \boldsymbol{s}^-, \boldsymbol{s}^+} \quad & z = \boldsymbol{e}\boldsymbol{s}^- + \boldsymbol{e}\boldsymbol{s}^+ \\
\text{s.t.} \quad & \boldsymbol{s}^- = \boldsymbol{x}_k - X\boldsymbol{\lambda} \\
& \boldsymbol{s}^+ = Y\boldsymbol{\lambda} - \boldsymbol{y}_k \\
& \boldsymbol{e}\boldsymbol{\lambda} = 1 \\
& \boldsymbol{\lambda} \geq 0, \boldsymbol{s}^- \geq 0, \boldsymbol{s}^+ \geq 0
\end{aligned}
$$

Other DEA-models
Additive model

- The Additive model arrives to a point in efficient frontier that is most distant from D
- The solution for D is
  $$s^{-*} = 1, s^{+*} = 1,$$
  $$\lambda_2^* = 1, \lambda_1^* = \lambda_3^* = \lambda_4^* = 0$$
- D would be projected on to B
- **Theorem:** DMU *k* is ADD-efficient if and only if it is BCC-efficient



$$\begin{aligned} \max_{\boldsymbol{\lambda}, \boldsymbol{s}^-, \boldsymbol{s}^+} \quad & z = \boldsymbol{e}\boldsymbol{s}^- + \boldsymbol{e}\boldsymbol{s}^+ \\ \text{s.t.} \quad & \boldsymbol{s}^- = \boldsymbol{x}_k - X\boldsymbol{\lambda} \\ & \boldsymbol{s}^+ = Y\boldsymbol{\lambda} - \boldsymbol{y}_k \\ & \boldsymbol{e}\boldsymbol{\lambda} = 1 \\ & \boldsymbol{\lambda} \geq 0, \boldsymbol{s}^- \geq 0, \boldsymbol{s}^+ \geq 0 \end{aligned}$$

Figure source: Cooper, W. W., Seiford, L. M., Tone, K., 2006: Introduction to Data Envelopment Analysis and Its Uses - With DEA-Solver Software and References, Springer Science+Business Media, New York.

Other DEA-models
Additive model, Property

- In some applications we would like to allow for negative data
- Additive model is **translation invariant**
- A DEA problem is translation invariant, if translating input and/or data results in a new problem that has the same optimal solution (for envelopment form)
- Note that *input*-oriented BCC model is translation invariant with respect to *outputs*

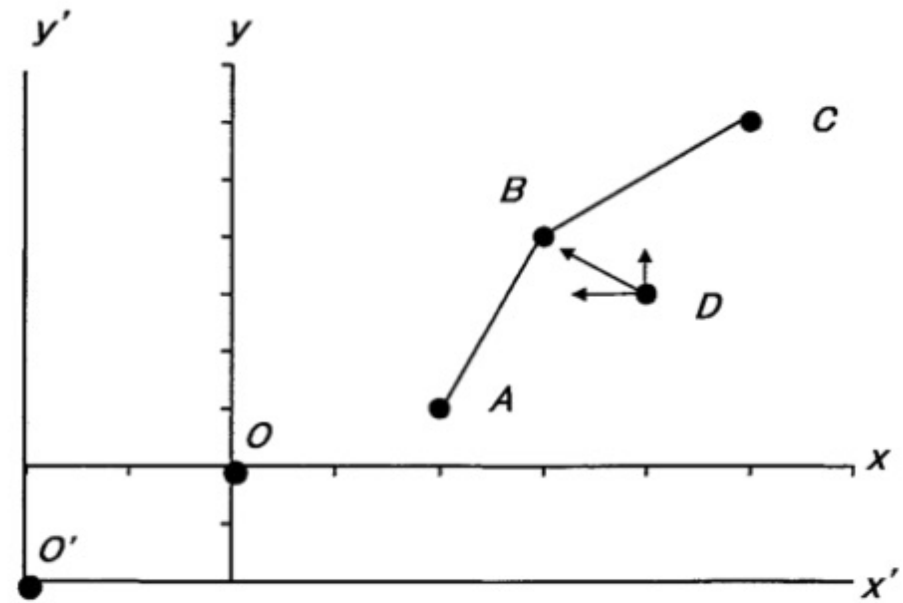


Figure source: Cooper, W. W., Seiford, L. M., Tone, K., 2006: Introduction to Data Envelopment Analysis and Its Uses - With DEA-Solver Software and References, Springer Science+Business Media, New York.

Other DEA-models
Introduction to SBM

- Finally, we would like a DEA-model that
  - Reflects all inefficiencies (technical and mix)
  - Has a single efficiency score
  - Considers both input and output side simultaneously
  - Is units invariant
- For these reasons, we augment the Additive model by introducing an efficiency score called the **Slacks-Based Measure of efficiency** (**SBM**)

Other DEA-models
SBM

$$\rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^- / x_{ik}}{1 + \frac{1}{s}\sum_{r=1}^{s} s_r^+ / y_{rk}}$$

- SBM ρ can be interpreted as the ratio of mean input and output mix inefficiencies
- ρ has the following properties
  - The measure is units invariant
  - The measure is monotone decreasing in each input and output slack
- We would like this ratio ρ to be minimized
  - SBM model

Other DEA-models
SBM model

- SBM model is the Additive model augmented with the minimization of ρ
- The resulting model is a fractional problem
- SBM has the properties we hoped for
  - Furthermore, $0 \leq \rho \leq 1$
- However, SBM is not translation invariant

$$
\begin{aligned}
\min_{\boldsymbol{\lambda}, \boldsymbol{s}^-, \boldsymbol{s}^+} \quad & \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^- / x_{ik}}{1 + \frac{1}{s}\sum_{r=1}^{s} s_r^+ / y_{rk}} \\
\text{s.t.} \quad & \boldsymbol{s}^- = \boldsymbol{x}_k - X\boldsymbol{\lambda} \\
& \boldsymbol{s}^+ = Y\boldsymbol{\lambda} - \boldsymbol{y}_k \\
& \boldsymbol{e}\boldsymbol{\lambda} = 1 \\
& \boldsymbol{\lambda} \geq 0, \boldsymbol{s}^- \geq 0, \boldsymbol{s}^+ \geq 0
\end{aligned}
$$

Other DEA-models
SBM model

- **Theorem:** DMU *k* is SBM-efficient if and only if $\rho^* = 1$ (slacks = 0)
- The solution for D is

$$s^{-*} = 1, s^{+*} = 1,$$

$$\lambda_2^* = 1, \lambda_1^* = \lambda_3^* = \lambda_4^* = 0,$$

$$\rho^* = \frac{1 - \frac{1}{1} \cdot 1/4}{1 + \frac{1}{1} \cdot 1/3} = 0.56$$

- **Theorem:** The optimal SBM ρ^* is not greater than the optimal CCR θ* (SBM accounts for *all* inefficiencies)

$$\begin{aligned}
\min_{\boldsymbol{\lambda}, \boldsymbol{s}^-, \boldsymbol{s}^+} \quad & \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} s_i^- / x_{ik}}{1 + \frac{1}{s}\sum_{r=1}^{s} s_r^+ / y_{rk}} \\
\text{s.t.} \quad & \boldsymbol{s}^- = \boldsymbol{x}_k - X\boldsymbol{\lambda} \\
& \boldsymbol{s}^+ = Y\boldsymbol{\lambda} - \boldsymbol{y}_k \\
& \boldsymbol{e}\boldsymbol{\lambda} = 1 \\
& \boldsymbol{\lambda} \geq 0, \boldsymbol{s}^- \geq 0, \boldsymbol{s}^+ \geq 0
\end{aligned}$$

Other DEA-models
Choosing the right DEA-model

- There are multiple DEA-models that excel in different situations
- Choosing the right model might involve considering
  - Input or output oriented or both?
  - Shape of the efficient frontier, returns-to-scale?
  - Technical efficiency or mix efficiency or both?
  - Negative data? Translation invariance
  - Units invariance?
- Often it is advisable to compare different models

References

- Cooper, W. W., Seiford, L. M., Tone, K., 2006: Introduction to Data Envelopment Analysis and Its Uses - With DEA-Solver Software and References, Springer Science+Business Media, New York.
- Doyle, J., Green, R., 1994. Efficiency and Cross-efficiency in DEA: Derivations, Meanings and Uses, Journal of the Operational Research Society 45/5, s. 567-578.

Homework

Output-oriented BCC-model

- In output-oriented BCC-model we maximize the outputs while using no more than the observed amount of any input
- **Task 1:** Solve the output-oriented BCC efficiency for DMU D from the whiteboard example
  - Note that $\eta_B^* \geq 1$
- **Task 2:** Solve the same problem for DMU D when the input values are subtracted by 3
  - That is, $X_- = \begin{pmatrix} -1 & 0 & 2 & 1 \end{pmatrix}$
- **Task 3:** Briefly comment on the results
- Submit to viljami.uusiharkala@aalto.fi by 9.00 19.11.2021

$$X = \begin{pmatrix} 2 & 3 & 5 & 4 \end{pmatrix},$$
$$Y = \begin{pmatrix} 1 & 4 & 6 & 3 \end{pmatrix}$$

$$\begin{aligned} \max_{\eta_B, \boldsymbol{\lambda}} \quad & \eta_B \\ \text{s.t.} \quad & \boldsymbol{x}_k \geq X\boldsymbol{\lambda} \\ & Y\boldsymbol{\lambda} \geq \eta_B \boldsymbol{y}_k \\ & \boldsymbol{e}\boldsymbol{\lambda} = 1 \\ & \boldsymbol{\lambda} \geq 0 \end{aligned}$$

Appendix
Properties of DEA-models

Model		CCR-I	CCR-O	BCC-I	BCC-O	ADD	SBM
Data	X	Semi-p	Semi-p	Semi-p	Free	Free	Semi-p
	Y	Free	Free	Free	Semi-p	Free	Free
Trans.	X	No	No	No	Yes	Yes[a]	No
Invariance	Y	No	No	Yes	No	Yes[a]	No
Units invariance		Yes	Yes	Yes	Yes	No	Yes
θ^*		[0, 1]	[0, 1]	(0, 1]	(0, 1]	No	[0, 1]
Tech. or Mix		Tech.	Tech.	Tech.	Tech.	Mix	Mix
Returns to Scale		CRS	CRS	VRS	VRS	C(V)RS[b]	C(V)RS

[a]: The Additive model is translation invariant only when the convexity constraint is added.
[b]: C(V)RS means Constant or Variable returns to scale according to whether or not the convexity constraint is included.

Table source: Cooper, W. W., Seiford, L. M., Tone, K., 2006: Introduction to Data Envelopment Analysis and Its Uses - With DEA-Solver Software and References, Springer Science+Business Media, New York.

Appendix
Linear formulation of SBM

- Define

$$\boldsymbol{S}^- = t\boldsymbol{s}^-,$$

$$\boldsymbol{S}^+ = t\boldsymbol{s}^+,$$

$$\boldsymbol{\lambda} = t\boldsymbol{\lambda},$$

$$\tau^* = \rho^*$$

$$\begin{aligned}
\min_{t,\boldsymbol{S}^-,\boldsymbol{S}^+,\boldsymbol{\Lambda}} \quad & \tau = t - \frac{1}{m}\sum_{i=1}^{m} S_i^- / x_{ik} \\
\text{s.t.} \quad & 1 = t + \frac{1}{s}\sum_{r=1}^{s} S_r^+ / y_{ik} \\
& \boldsymbol{S}^- = t\boldsymbol{x}_k - X\boldsymbol{\Lambda} \\
& \boldsymbol{S}^+ = Y\boldsymbol{\Lambda} - t\boldsymbol{y}_k \\
& \boldsymbol{\Lambda} \geq 0, \boldsymbol{S}^- \geq 0, \boldsymbol{S}^+ \geq 0, t \geq 0
\end{aligned}$$