Aalto University
School of Science

REA method: Preference Programming in efficiency analysis

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Contents

Notation

K **DMUs**

M **Inputs**

N **Outputs**

$\boldsymbol{x_k} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_m})$ **Vector of inputs for DMU$_k$**

$\boldsymbol{y_k} = (\boldsymbol{y_1}, \boldsymbol{y_2}, \dots, \boldsymbol{y_n})$ **Vector of outputs for DMU$_k$**

$\boldsymbol{v} = (\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_m})$ **Weight vector for the inputs**

$\boldsymbol{u} = (\boldsymbol{u_1}, \boldsymbol{u_2}, \dots, \boldsymbol{u_n})$ **Weight vector for the outputs**

CCR-DEA

Maximize the efficiency for each DMU by choosing the most favourable input and output weights.

The efficiency of DMU_k is calculated as so:

$$E_k(u,v) = \frac{output}{input} = \frac{u^T y_k}{v^T x_k} = \frac{\sum_{i=1}^{n} v_i y_{ik}}{\sum_{i=1}^{m} u_i x_{ik}}$$

Shortcomings of CCR-DEA

- **Other combinations of input/output weights are neglected**
  - Relevant preference information is lost

- **Method is sensitive to outliers**
  - An outlier DMU can change the efficient frontier drastically

- **Requires a large number of DMUs to form the efficient frontier accurately**

Salo and Punkka (2011)

Ratio-based efficiency analysis (REA)

- **Pairwise comparisons between DMUs using ALL feasible sets of weights**

- **Important concepts:**
  - Ranking intervals
  - Efficiency dominance
  - Efficiency bounds

Ranking intervals

$$E_k(u, v) = \frac{u^T y_k}{v^T x_k}$$

- **DMUs are ranked by their efficiency scores for all feasible weights**

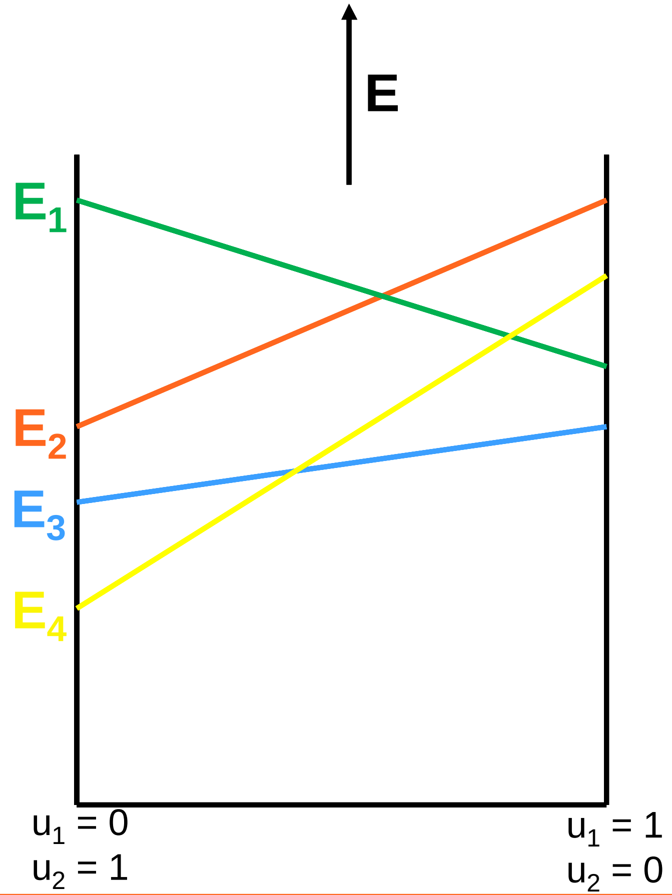
- **What are the best and worst ranks a given DMU can get when the weights are varied?**
  - Can be solved with linear optimization

A1

Ranking intervals

- **4 DMUs, one input, two outputs**
- **Output weights u_1 u_2 are varied**

DMU	1	2	3	4
Max rank	1	1	3	2
Min rank	3	2	4	4
Interval	[1,3]	[1,2]	[3,4]	[2,4]

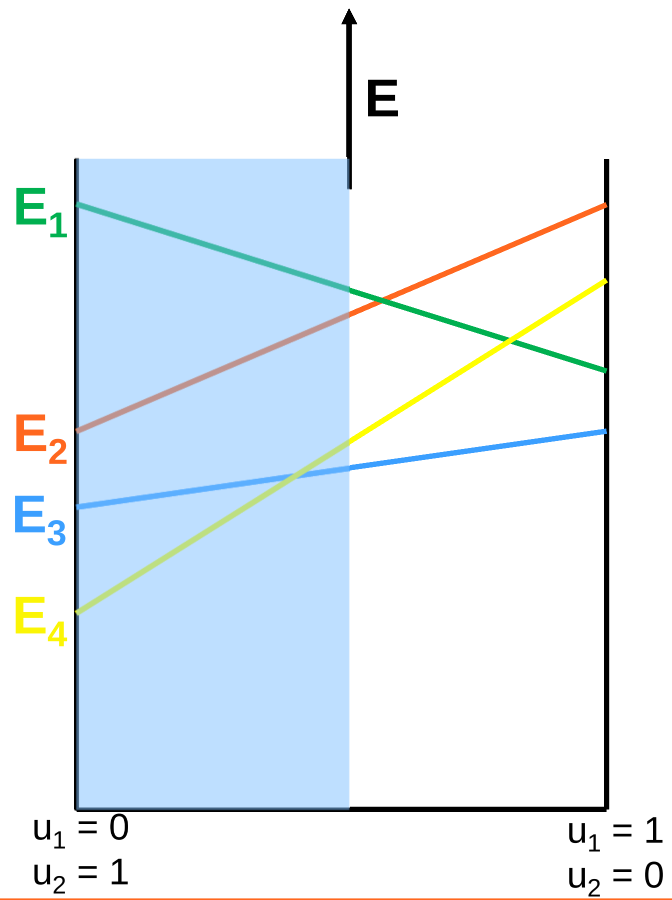


A1 Author; 15.11.2021

Ranking intervals

- **Additional preference information narrows the intervals**
- **If $u_1 < u_2$ we are in the blue region**

DMU	1	2	3	4
Max rank	1	2	3	3
Min rank	1	2	4	4
Interval	[1,1]	[2,2]	[3,4]	[3,4]



Ranking intervals

r_k^{min}:

$$min_{x,y,z}\ 1 + \sum_{l \neq k} z_l$$

$$\sum_n u_n y_{nl} \leq \sum_m v_m x_{ml} + C z_l$$

$$l \in \{1, \ldots, k\}, \mathrm{l} \neq k$$

$$\sum_n u_n y_{nk} = \sum_m v_m x_{mk}$$

$$z_l \in \{0, 1\}, l \neq k$$

$$(u, v) \in (S_u, S_v)$$

r_k^{max}:

$$max_{x,y,z}\ 1 + \sum_{l \neq k} z_l$$

$$\sum_n v_m x_{ml} \leq \sum_m u_n y_{nl} + C(1 - z_l)$$

$$l \in \{1, \ldots, k\}, \mathrm{l} \neq k$$

$$\sum_n u_n y_{nk} = \sum_m v_m x_{mk}$$

$$z_l \in \{0, 1\}, l \neq k$$

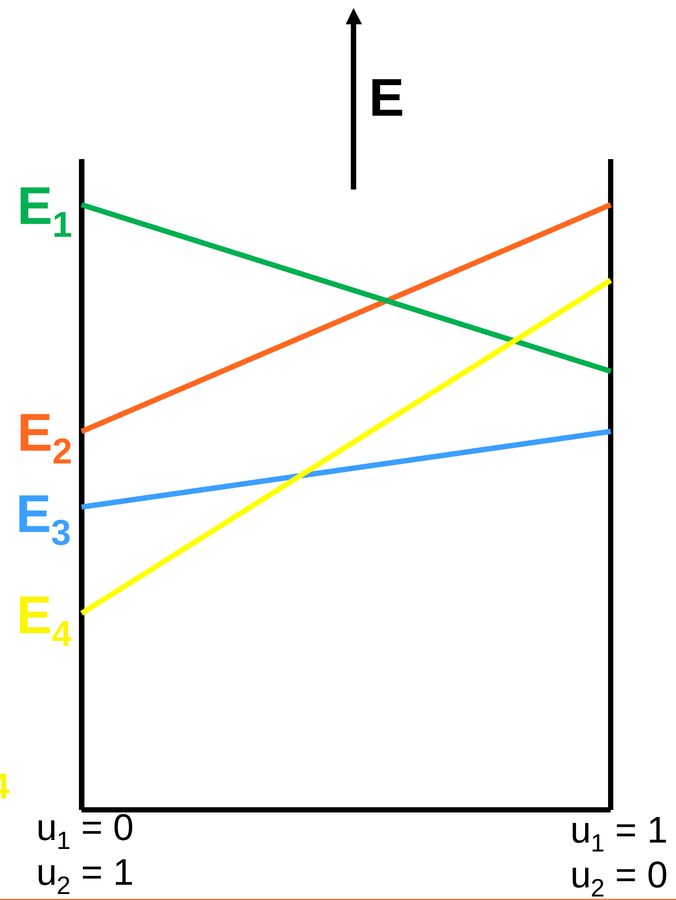
$$(u, v) \in (S_u, S_v)$$

Efficiency dominance

- **DMU_k dominates DMU_l if**

1. $E_k(\mathbf{u},\mathbf{v}) \geq E_l(\mathbf{u},\mathbf{v})$ for all feasible **u** and **v**

2. $E_k(\mathbf{u},\mathbf{v}) > E_l(\mathbf{u},\mathbf{v})$ for some feasible **u** and **v**



DMU_2 *dominates* DMU_3 and DMU_4

Efficiency dominance

- **The pairwise dominance relation is defined as so**

$$D_{k,l}(\mathbf{u},\mathbf{v}) = \frac{E_k(\boldsymbol{u},\boldsymbol{v})}{E_l(\boldsymbol{u},\boldsymbol{v})}$$

- **If $min\, D_{k,l}(u,v) \geq 1$ and $max\, D_{k,l}(u,v) > 1$, DMU$_k$ dominates DMU$_l$**

- **Can be solved with linear programming**

Efficiency dominance

$$D_{k,l}(u,v) = \frac{E_k(u,v)}{E_l(u,v)}$$

$$= \frac{\sum_n u_n y_{nk}}{\sum_m v_m x_{mk}} \Big/ \frac{\sum_n u_n y_{nl}}{\sum_m v_m x_{ml}}$$

$$min_{u,v} \sum_n u_n y_{nk}$$

$$\sum_n u_n y_{nl} = \sum_m v_m x_{ml}$$

$$\sum_m v_m x_{mk} = 1$$

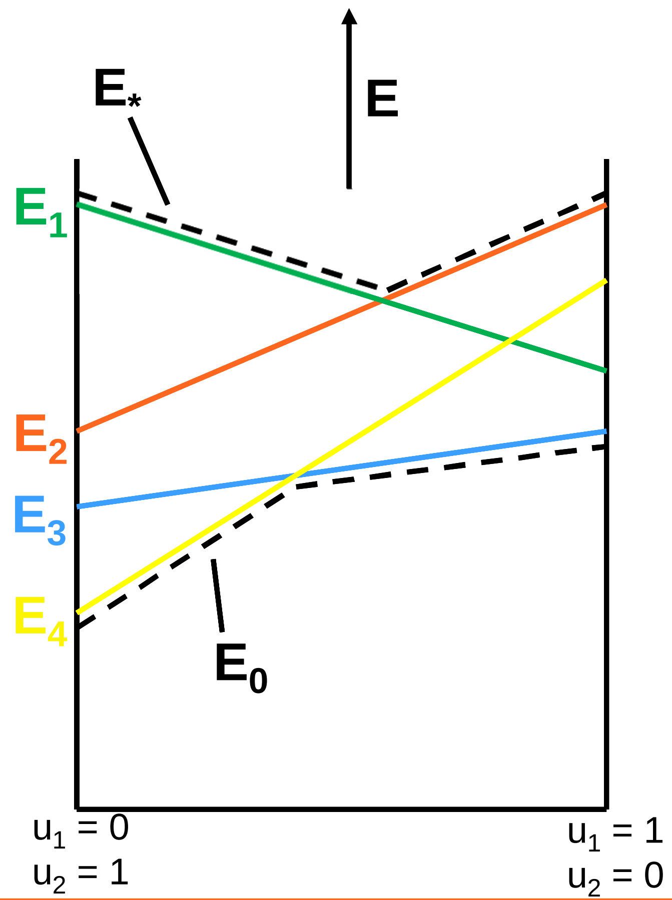
$$(u,v) \in (S_u, S_v)$$

If the solution is exactly 1, the same problem is maximized. If that solution is >1, we have dominance.

Efficiency bounds

- **How efficient can a given DMU be with respect to the most (E_*) and least (E_0) efficient DMUs?**
- **Another optimization problem over u,v.**

$E_2 \in [0.6, 1.0] E_*$
$E_2 \in [1.3, 1.8] E_0$



Efficiency bounds

$$D_{k,\bar{L}}(u,v) = \frac{E_k(u,v)}{max_{l \in L} E_l(u,v)} \qquad D_{k,\underline{L}}(u,v) = \frac{E_k(u,v)}{min_{l \in L} E_l(u,v)}$$

$\underline{D}_{k,\bar{L}}$:

$$max_{u,v} \sum_n u_n y_{nk}$$

$$\sum_m v_m x_{mk} = 1, \qquad (u,v) \in (S_u, S_v)$$

$$\sum_n u_n y_{nl} \le \sum_m v_m x_{ml}, \qquad l \in L$$

$\bar{D}_{k,,\underline{L}}$:

$$min_{u,v} \sum_n u_n y_{nk}$$

$$\sum_m v_m x_{mk} = 1, \qquad (u,v) \in (S_u, S_v)$$

$$\sum_n u_n y_{nl} \le \sum_m v_m x_{ml}, \qquad l \in L$$

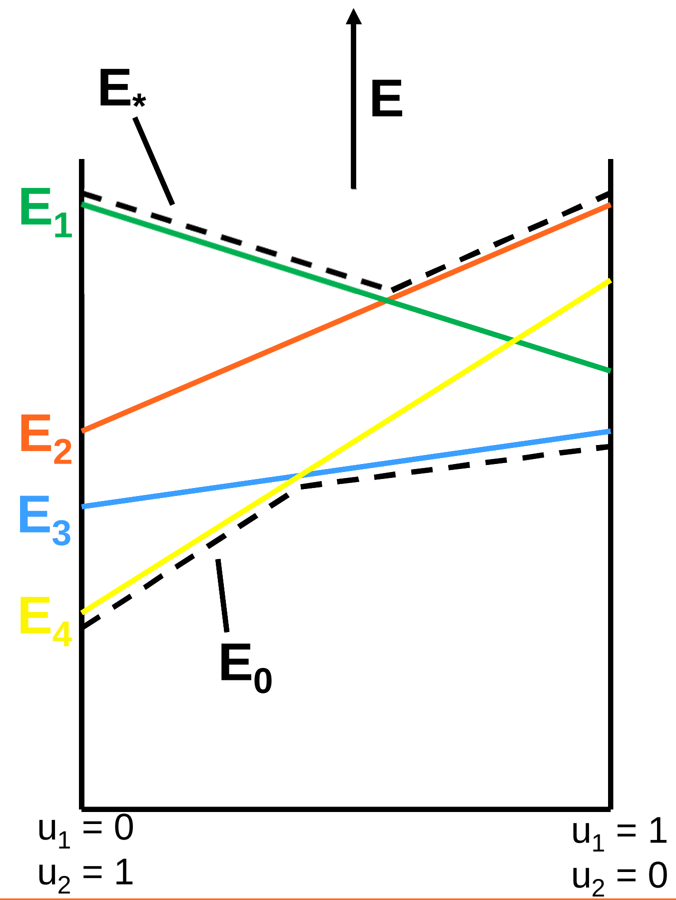
Efficiency bounds

Recall: $E_2 \in [0.6, 1.0] E_*$
$E_2 \in [1.3, 1.8] E_0$

In this case

$\overline{D}_{2,\overline{L}} = 1.0$ $\quad \underline{D}_{2,\overline{L}} = 0.6$

$\overline{D}_{2,\underline{L}} = 1.8$ $\quad \underline{D}_{2,\underline{L}} = 1.3$



Applying REA to Scottish Health Boards

Schang et. al. (2016)

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Application

- **Health care quality of the 14 boards are compared**
- **Inputs and outputs are normalized to [0,1] intervals.**
- **Difficulties in defining weights → Need for robust ranking methods, aka. REA**

Inputs	**Outputs**
- Resident population	**- Over 18 week wait from referral to treatment**
- Number of occupied bed days (OBD)	**- Longer than 4 hour wait in A&E**
	- Emergency admission
	- Delayed discharge
	- C.difficile infection
	- MRSA/MSSA infection

Application

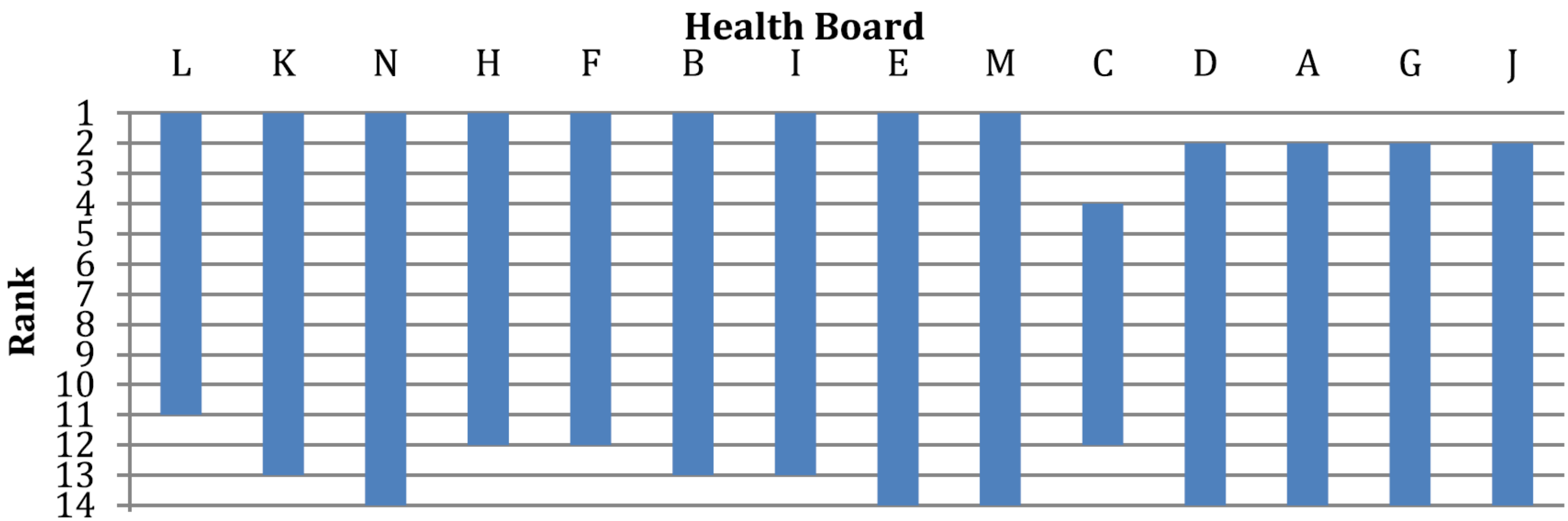
		4-hour A&E waiting	Emergency admissions	MRSA/MSSA
A	Ayrshire & Arran	8312	3646	23
B	Borders	3267	3612	21
C	Dumfries & Galloway	5987	3130	27
D	Fife	4559	2725	35
E	Forth Valley	8238	2513	26
F	Grampian	3812	2239	25
G	Greater Glasgow & Clyde	6956	3061	34
H	Highland	2199	2825	17
I	Lanarkshire	8667	2671	24
J	Lothian	9172	2495	30
K	Orkney	1663	2661	9
L	Shetland	730	2555	13
M	Tayside	1119	2964	36
N	Western Isles	1666	3320	4

...

Application

- **Ranking intervals and dominances were solved with three different sets of weights. Input variables were unrestricted in each scenario.**
  - **Set 1: No weight restrictions**
  - **Set 2: Ordinal weights restrictions**
  - **Set 3: Ordinal and proportional weight restrictions**

No weighting – Ranking intervals



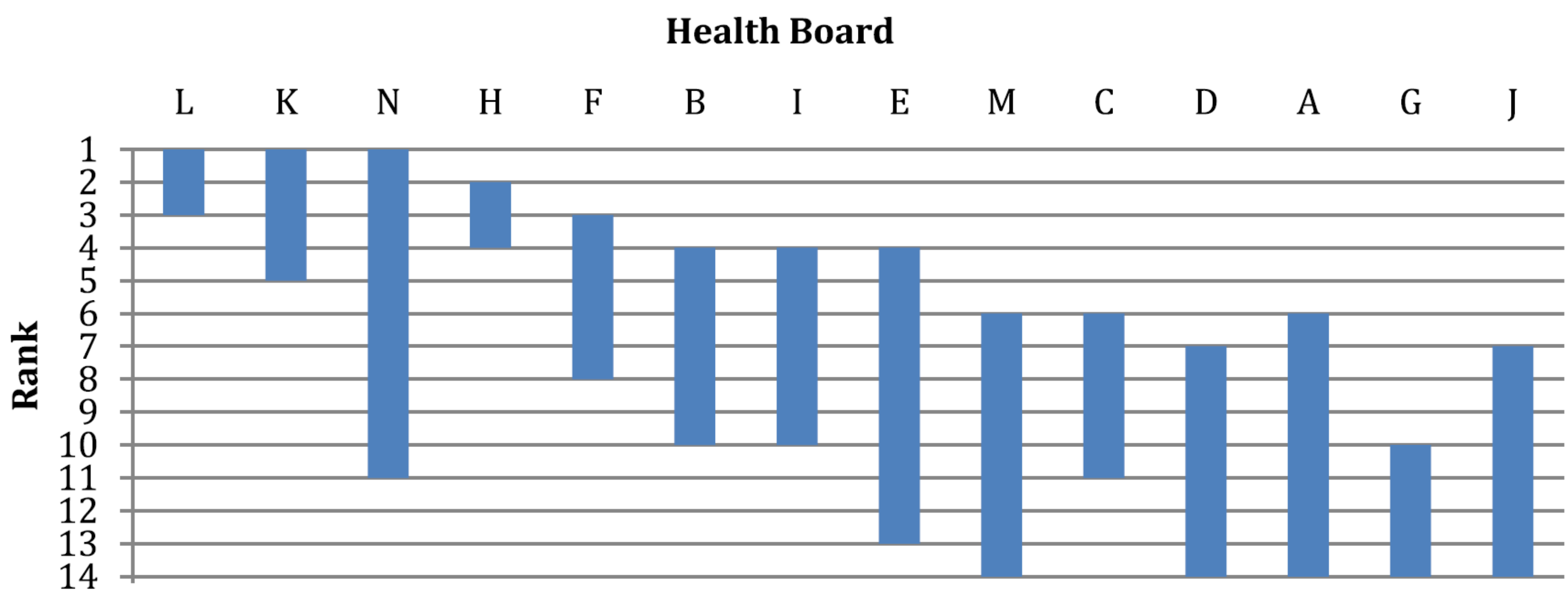
Schang et al. (2016)

Ordinal weighting

DMs were asked to rank the outputs from worst to least bad

1. an MRSA/MSSA infection
2. an emergengy admission
3. a C.difficile infection
4. longer than 18 week wait from referral to treatment
5. longer than 4 hour wait in A&E
6. a delayed discharge

Ordinal weighting – Rank intervals



Schang et al. (2016)

Proportional weighting

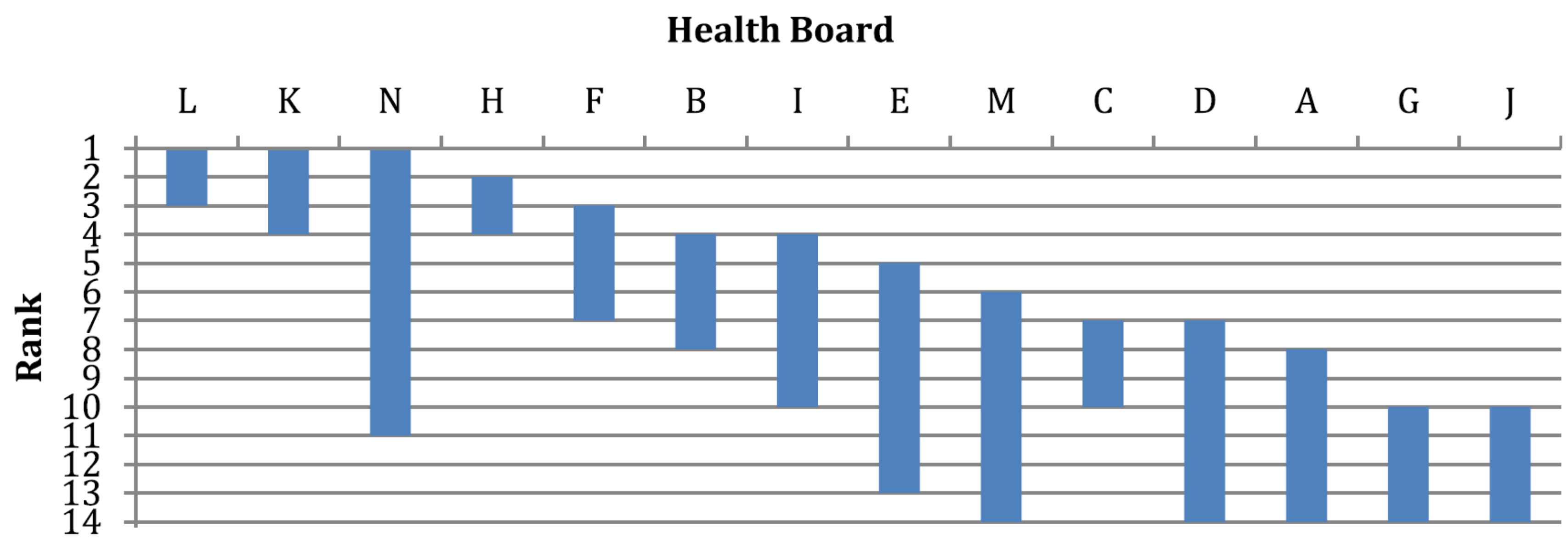
” Avoiding an event of the worst healthcare quality measure cannot be more than ten times as valuable as avoiding an event of the least bad quality measure”

$$u_n \geq 0.1 \;\forall\, n$$

“Avoiding C.difficile infection must be at least ¼ as valuable as avoiding MRSA/MSSA infection”

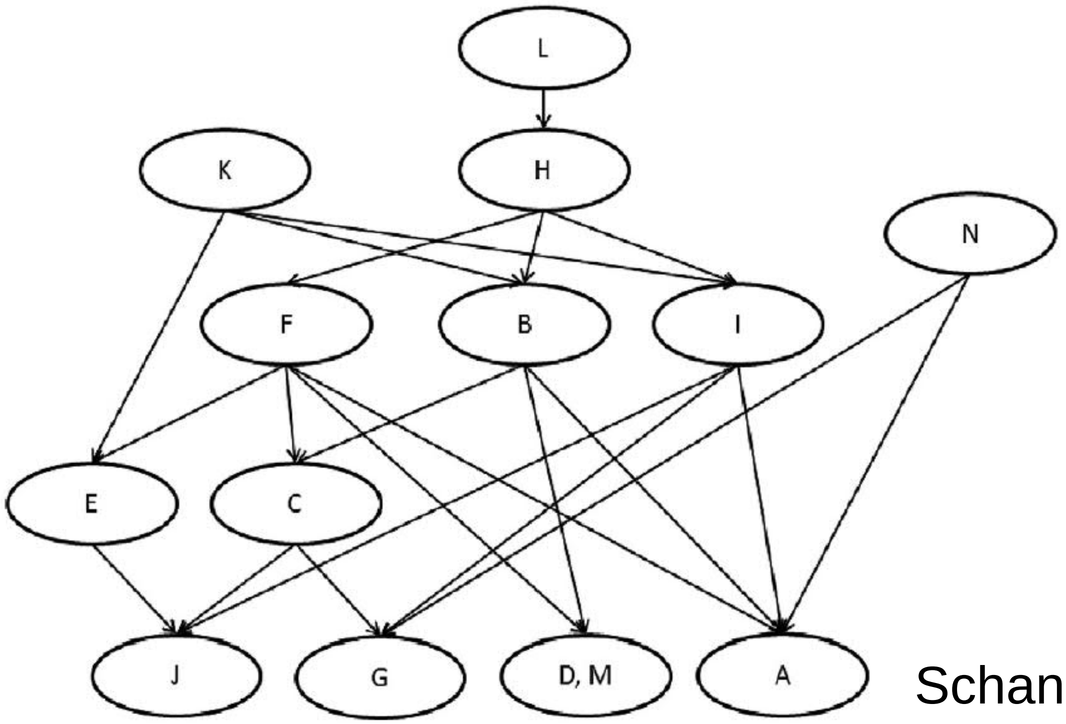
$$u_3 \geq \frac{1}{4} u_1$$

Ordinal & Proportional weighting – Rank intervals



Schang et al. (2016)

Ordinal & Proportional weighing - Dominances



Schang et al. (2016)

Homework

Are the following statements true or false? Justify your answers.

a) **DMU_k can not dominate DMU_l if their ranking intervals overlap.**
b) **If DMU_k is dominated by n DMUs. The highest rank DMU_k can possibly have for some u,v is n+1.**
c) **It always holds that $\overline{D}_{k,\overline{L}} \leq \underline{D}_{k,\underline{L}}$**
d) **There can not be any dominances if weights are non-restricted.**

Send the answers to lauri.vaara@aalto.fi
DL: 26.11 at 9:00

References

[1] Salo, A., Punkka, A., 2011: Ranking Intervals and Dominance Relations for Ratio-Based Efficiency Analysis, Management Science 57/1, s. 200-214

[2] Schang, L., Hynninen, Y., Morton, A., Salo, A., 2016: Developing robust composite measures of healthcare quality - Ranking intervals and dominance relations for Scottish Health Boards, Social Science & Medicine 162, s. 59-67