Aalto University
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Additive-linear portfolio value function: theory and an application

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Idea: Selecting a portfolio of research projects

- **Have a set of potential projects $P = \{P_1, P_2, P_3\}$**
- **Need to select an "optimal" portfolio x. Modeled as a matrix!**
- **The quality of each project is measured using several attribute criterion*. E.g**

$$\begin{aligned} P_1 &= (x_{11}, x_{12}, x_{13}) \\ P_2 &= (x_{21}, x_{22}, x_{23}) \\ P_3 &= (x_{31}, x_{32}, x_{33}) \end{aligned} \longrightarrow \mathrm{x} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

- **Our notation comes mostly from Liesiö (2014)**

Idea: Selecting a portfolio of research projects

$$\mathbf{x} = \begin{pmatrix} \boxed{\begin{matrix} x_{11} \\ x_{21} \\ x_{31} \end{matrix}} & \boxed{\begin{matrix} x_{12} \\ x_{22} \\ x_{32} \end{matrix}} & \boxed{\begin{matrix} x_{13} \\ x_{23} \\ x_{33} \end{matrix}} \end{pmatrix}$$

Performance Potential

Potential for low cost

Experience of Team

Idea: Selecting a portfolio of research projects

- As usual, attributes are assumed to have scalar most preferred levels x_i^* and a least preferred levels x_i^0
- So in this case,

$$x_i^0 \leq x_{ji} \leq x_i^*$$

for each attribute i and project j.

The additive-linear portfolio value function

- **Just to keep the notion clear let's derive what an additive-linear value function should be for the project selection problem. (see the whiteboard now!)**

Something odd going on?

- **We are *choosing* projects to put inside our portfolio**
- **But the portfolio matrix seems to already include all projects?**

Something odd going on?

- **We are *choosing* projects to put inside our portfolio**
- **But the portfolio matrix seems to already include all projects?**
- **Trick: Assume unwanted projects have the worst possible attributes.**
  - What does that mean?
  - Back to the whiteboard.

Questions

- **How much must we assume about DMs preferences for such a function to exist? (Our main focus)**
- **What procedures are available to elicit said function?**
  - Standard trade off techniques apply, see Liesiö (2014) for example.

Assumptions – setting things up

- **Recall our portfolio matrix of projects x.**
- **We assume DMs preferences are given by a (weak) order on the space of all possible portfolios X denoted by $\preccurlyeq$.**
- **Note: X is the space of $m \times n$ matrices whose rows are the attribute m-vectors of each of the n projects.**
- **And also another (weak) order on $X \times X$ denoted by $\succcurlyeq_D$**

***The following exposition largely follows Liesiö (2014)**

Assumptions – setting things up

- $x_1 \preccurlyeq x_2$ iff $V(x_1) \leq V(x_2)$
- $x_1 \leftarrow x_2 \preccurlyeq x_3 \leftarrow x_4$ iff $V(x_1) \leftarrow V(x_2) \leq V(x_3) \leftarrow V(x_4)$
- For matrices $x_1, x_2, x_3, x_4 \in X$
- A value function representing a preference in this way is often said to be *measurable* (see Dyer and Sarin, 1979).
- GOAL: Show that these axioms + four further assumptions forces the value function to be additive-linear (and vice versa).
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Note for clarity

- **The number of attributes of our additive-linear portfolio value function is NOT $n = 3$!**
- **Since it depends on the entire portfolio, there are actually $m \times n = 3$ attributes!**
- **We will often refer to X_{ji} to refer to the set of a particular attribute.**
- **Example: X_{31} is project 3's attribute number 1.**
- **Attributes refer to any set X_{ji}**

Assumption 1 – Project symmetry

- **Intuition: we are not biased for one project or the other. If Project 1 contains the same attribute values as Project 2, we have equal preference.**
- **So,**

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Difference independence

$$\begin{pmatrix} \mathbf{10} & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \leftarrow \begin{pmatrix} \mathbf{1} & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\sim$$

$$\begin{pmatrix} \mathbf{10} & 4 & 3 \\ 11 & 12 & 12 \\ 111 & 22 & 23 \end{pmatrix} \leftarrow \begin{pmatrix} \mathbf{1} & 11 & 11 \\ 11 & 21 & 31 \\ 11 & 21 & 13 \end{pmatrix}$$

Attribute X_{11} is DI of other attributes.

Weak Difference Independence

If

$$\begin{pmatrix} \mathbf{10} & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \leftarrow \begin{pmatrix} \mathbf{1} & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\succeq$$

$$\begin{pmatrix} \mathbf{10} & 4 & 3 \\ 11 & 12 & 12 \\ 111 & 22 & 23 \end{pmatrix} \leftarrow \begin{pmatrix} \mathbf{1} & 11 & 11 \\ 11 & 21 & 31 \\ 11 & 21 & 13 \end{pmatrix}$$

then…

Weak Difference Independence

If

$$\begin{pmatrix} \mathbf{10} & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \leftarrow \begin{pmatrix} \mathbf{1} & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\succeq$$

$$\begin{pmatrix} \mathbf{10} & 4 & 3 \\ 11 & 12 & 12 \\ 111 & 22 & 23 \end{pmatrix} \leftarrow \begin{pmatrix} \mathbf{1} & 11 & 11 \\ 11 & 21 & 31 \\ 11 & 21 & 13 \end{pmatrix}$$

then…

Weak Difference Independence

$$\begin{pmatrix} \mathbf{10} & 0 & 1 \\ 0 & 122 & 0 \\ 1 & 1 & 1 \end{pmatrix} \leftarrow \begin{pmatrix} \mathbf{1} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\succeq$$

still

$$\begin{pmatrix} \mathbf{10} & 10 & 11 \\ 10 & 122 & 10 \\ 11 & 11 & 11 \end{pmatrix} \leftarrow \begin{pmatrix} \mathbf{1} & 10 & 11 \\ 10 & 11 & 10 \\ 11 & 10 & 11 \end{pmatrix}$$

***We can change the other values to whatever.**

Assumption 2 – Weak Difference Independence

- **Each X_{ji} should be weak difference independent of the other attributes.**

Assumption 3 – WDI Part 2

- **Attributes for a particular project-level criterion are weak difference independent from all other attributes**
  - What?

Assumption 3 – WDI Part 2

- **Attributes for a particular project-level criterion are weak difference independent from all other attributes**
  - What?

Assumption 4 - DI

- **Each set of attributes for a particular project *j* are difference independent of all other attributes.**

- **If previous conditions are satisfied then the DMs value function must be additive-linear (and vice versa):**

$$V(x) = \sum_{j=1}^{m} w_i \sum_{i=1}^{n} v_i(x_{ji})$$

Golabi et al. (1981)

SELECTING A PORTFOLIO OF SOLAR ENERGY PROJECTS USING MULTIATTRIBUTE PREFERENCE THEORY*

KAMAL GOLABI†, CRAIG W. KIRKWOOD† AND ALAN SICHERMAN†

This article reports a procedure developed to assist the U.S. Department of Energy in selecting a portfolio of solar energy applications experiments. The procedure has also been used in other government procurements and appears to be applicable in a variety of project funding processes. The technical quality of each proposed applications experiment was summarized through the use of multiple evaluation measures, or attributes. These were combined into a single index of the overall technical quality of an experiment through the use of a multiattribute utility function. Recently derived results in measurable value theory were applied to derive an index of the overall technical quality of a portfolio of experiments. Budgetary and programmatic issues were handled through the use of constraints. This approach allowed the portfolio selection problem to be formulated as an integer linear program. Details of the application are presented, including a disucssion of the data requirements and assessment procedure used. The portfolio selection procedure was successfully applied, and variations of it have been successfully used in four other solar energy procurements.

(RESEARCH AND DEVELOPMENT—PROJECT SELECTION; UTILITY/PREFERENCE—MULTIATTRIBUTE: PROGRAMMING—INTEGER, APPLICATIONS)

1. Introduction

- Used additive-linear model in selecting research project portfolio for US Dept. Energy.
- 22 selection criteria (attributes)
- Solved using ILP
- Watch out for the notation.

Nice connection to utility theory - Dyer & Sarin, (1979)

+ Some linearity utility assumptions (holds in Golabi's case)

Suppose we have assessed the additive value function $\mathring{v}$ and wish to obtain either v or u. We can use the results of the following theorem and its corollary.

THEOREM 5.

(A) *Either*

1. $\mathring{v}(x) = v(x)$ *and* $\mathring{v}_i(x_i) = v_i(x_i)$, $i = 1, \ldots, n$, *or*
2. $\mathring{v}(x)\ ln(1 + \lambda) = ln[1 + \lambda v(x)]$ *and* $\mathring{\lambda}_i \mathring{v}_i(x_i)\ ln(1 + \lambda) = ln[1 + \lambda\lambda_i v_i(x_i)]$, $i = 1, \ldots, n$.

(B) *Either*

1. $\mathring{v}(x) = u(x)$ *and* $\mathring{v}_i(x_i) = u_i(x_i)$, $i = 1, \ldots, n$, *or*
2. $\mathring{v}(x)\ ln(1 + k) = ln[1 + ku(x)]$ *and* $\mathring{\lambda}_i \mathring{v}_i(x_i)\ ln(1 + k) = ln[1 + kk_i u_i(x_i)]$, $i = 1, \ldots, n$.

Proof. We present the proof for part A only since the logic of the proof

Golabi et al. (1981)

- **Used to use utility elicitation techniques to elicit the value function.**
- **Apparently, they found direct value function methods difficult.**

References

- ***Measurable Multiattribute Value Functions for Portfolio Decision Analysis,*** **Juuso Liesiö** ***(2014)***
- ***Selecting a Portfolio of Solar Energy Projects Using Multiattribute Preference Theory,*** **Golabi et al. (1981)**
- ***Measurable Multiattribute Value Functions,*** **Dyer and Sarin (1979)**

Homework

- **Describe some of the potential limitations of the additive linear model. For example describe when it might not be an appropriate model for a DMs preferences. (Nothing formal required, just basic ideas).**
  - Liesiö (2014) would be a good start. Specifically, the introduction for the type of thinking that might be useful/references to examples to look at.
- **Email: thomas-roy.holt@aalto.fi (By 9.00, Oct 1)**
- **Try to use subject: ”additive-linear hw” if possible.**