Quantum Information   Spring 2021   Exam 31.5.2021

Solutions are due on Monday May 31, 18:00.

Problem 1

a)

Let $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ in the Hilbert space $\mathbb{C}^2$. Calculate

$$HZH\,|0\rangle \quad \text{and} \quad HZH\,|1\rangle \ ,$$

where H is the Hadamard transform. The unitary transform H is defined by

$$H\,|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^k\,|1\rangle) \ , \quad k \in \{0,1\} \ .$$

b)

Calculate

$$(H \otimes H)U_{CNOT}(H \otimes H)\,|j,k\rangle$$

where $|j,k\rangle \equiv |j\rangle \otimes |k\rangle$ with $j,k \in \{0,1\}$, and the answer is in form of a ket $|m,n\rangle$ where $m,n \in \{0,1\}$. The controlled-NOT is defined

$$U_{CNOT} \equiv |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes X \ ,$$

where $\mathbb{I}$ is the 2×2-identity matrix and $X = |0\rangle\langle 1| + |1\rangle\langle 0|$.

Problem 2

We have a two-qubit density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{H}_A \otimes \mathcal{H}_B \ .$$

a)

Is the state ρ pure or not?

b)

Calculate the reduced density matrix

$$\rho_A = \mathrm{tr}_{\mathcal{H}_B} \ .$$

c)

Calculate the entanglement entropy

$$S(\rho_A) = -\operatorname{tr}\left(\rho_A \log_2 \rho_A\right) .$$

Are the two qubits entangled?

Problem 3

The amplitude damping channel on one qubit has an operator-sum representation with the operation elements

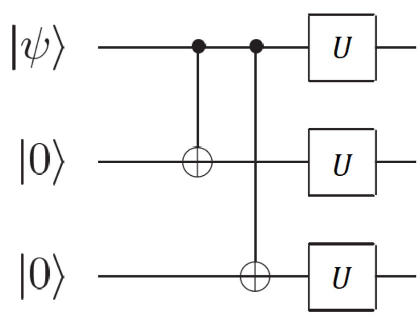
$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} , \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} ,$$

where $0 < \gamma < 1$. Show that the amplitude damping channel transforms the Bloch sphere vector $\vec{r}$ of the qubit as

$$(r_x, r_y, r_z) \mapsto \left(r_x\sqrt{1-\gamma}, r_y\sqrt{1-\gamma}, \gamma + r_z(1-\gamma)\right) .$$

Problem 4

The following figure depicts an encoding circuit for a particular quantum error correction code, where U is some fixed 1-qubit unitary.



- **a)** What kind of errors does this quantum error correction code protect against? Verify explicitly by checking that the quantum error correction condition is satisfied.
- **b)** How should the syndrome extraction be implemented, and how is the error identified based on this?
- **c)** What is the distance of this code? Explain.

Problem 5

Let σ_2 be the second Pauli matrix. Then

$$\sigma_2 \otimes \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} .$$

a)

Find the normalized state ($\gamma \in \mathbb{R}$)

$$|\psi\rangle = e^{i\gamma\sigma_2 \otimes \sigma_2} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv e^{i\gamma\sigma_2 \otimes \sigma_2} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) .$$

b)

Find the values of γ such that $|\psi\rangle$ is a product state.