Aalto University
School of Electrical Engineering

ELEC-C9610 Basics in Electronics

Lecture 3: Circuit analysis methods

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Systematic circuit analysis methods: mesh current and node voltage method

- If a circuit is big, it is not reasonble to use circuit transforms.
- Without systematic approach, it is difficult to write exactly correct number of voltage-current-equations so that a whole problem is determined.
- Matrices are efficient way to handle large systems of equations and solve them.

In systematic methods, used variables are mesh currents and node voltages. The real currents and voltages of the circuit are determined using these auxiliary variables.

Learn matrices in one slide!

Example:

$$\underbrace{\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}}_{2\times\color{magenta}{2}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}}_{\color{magenta}{2}\times 1} = \underbrace{\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}}_{2\times 1}$$

Terminology: rows and columns, diagonal

Multiplication of matrices: 'row times column'

The matrix equation above is another way to write this pair of equations:

$$\begin{cases} R_{11}I_1 + R_{12}I_2 = E_1 \\ R_{21}I_1 + R_{22}I_2 = E_2 \end{cases}$$

Determinant (2x2 matrix):

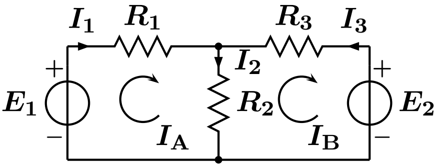
$$\Delta = \begin{vmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{vmatrix} = R_{11}R_{22} - R_{12}R_{21}$$

Mesh current method

In mesh current method, we write the minimum number of Kirchhoff's voltage equation at the systematic manner as a matrix form.

- Equations should be written as many as there are unknowns in the circuit. In practice, number of mesh currents is number of 'windows' in the circuit, if the circuit can be drawn to a plane.
- Mesh currents can be chosen freely, but some of the mesh currents should go through every branch of the circuits.
- The real currents of the circuit are given with mesh currents. Mesh current and real currents might easily be mixed if you are not careful with notations.
- In mesh current method, only voltage sources are allowed so possible current sources should transforms to voltage sources.

Idea of mesh current method



$$I_1 = I_A$$
$$I_2 = I_A - I_B$$
$$I_3 = -I_B$$

Write voltage equations using Kirchhoff's law:

$$\begin{cases} R_1 I_1 + R_2 I_2 = E_1 \\ R_3 I_3 + R_2 I_2 = E_2 \end{cases}$$

Substitute the mesh currents to the equations:

$$\begin{cases} (R_1 + R_2) I_A - R_2 I_B = E_1 \\ R_2 I_A - (R_2 + R_3) I_B = E_2 \end{cases}$$

Same with a matrix: signs of the lower equation were changed

$$\begin{bmatrix} R_1 + R_2 & -R_2 \\ -R_2 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} I_A \\ I_B \end{bmatrix} = \begin{bmatrix} E_1 \\ -E_2 \end{bmatrix}$$

Mesh current method - writing the matrix equation

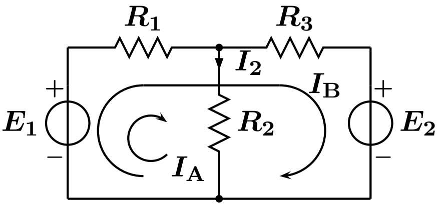
Matrix equation: $\boxed{\boldsymbol{RI} = \boldsymbol{E}}$

- diagonal elements of $\boldsymbol{R}$ matrix are the summation of resistances in each loop
- Off-diagonal elements are common resistances of corresponding loop taking the direction into account with sign (currents to opposite directions $\Rightarrow -$).
- The direction of sources are taken into account with sign (from $- \ \rightarrow + \ \Rightarrow +$).

Matrix is symmetric, i.e. for any $\boldsymbol{i}$ and $\boldsymbol{j}$ $\boldsymbol{R_{ij}} = \boldsymbol{R_{ji}}$.

Example on mesh current method

We choose this time mesh currents in another way. Now we write equations directly to the matrix form.



$$\begin{bmatrix} R_1 + R_2 & R_1 \\ R_1 & R_1 + R_3 \end{bmatrix} \begin{bmatrix} I_A \\ I_B \end{bmatrix} = \begin{bmatrix} E_1 \\ E_1 - E_2 \end{bmatrix}$$

How do we solve for example current I_2?

Cramer's rule

One unknown can be easily solved with Cramer's rule.

Cramer's rule: When solving $\boldsymbol{k}$th unknown, the $\boldsymbol{k}$th column vector of a $\boldsymbol{R}$ matrix is replaced with a source vector $\boldsymbol{E}$ for which we calculated the determinant, which is then divided by the determinant $\boldsymbol{\Delta}$ of the original matrix $\boldsymbol{R}$.

Deriving the Cramer's rule is based on the properties of the determinant.

Cramer's rule for our example

$$\begin{bmatrix} R_1 + R_2 & R_1 \\ R_1 & R_1 + R_3 \end{bmatrix} \begin{bmatrix} I_A \\ I_B \end{bmatrix} = \begin{bmatrix} E_1 \\ E_1 - E_2 \end{bmatrix}$$

$$\begin{aligned} I_2 = I_A &= \frac{\begin{vmatrix} E_1 & R_1 \\ E_1 - E_2 & R_1 + R_3 \end{vmatrix}}{\begin{vmatrix} R_1 + R_2 & R_1 \\ R_1 & R_1 + R_3 \end{vmatrix}} \\ &= \frac{E_1(R_1 + R_3) - (E_1 - E_2)R_1}{(R_1 + R_2)(R_1 + R_3) - R_1^2} \\ &= \frac{E_1 R_3 + E_2 R_1}{R_1 R_3 + R_2(R_1 + R_3)} \end{aligned}$$

Choosing mesh currents at different way creates a different matrix and intermediate phases change, but the final result is same.

Node voltage method

- In node voltage method, the variables are voltages between nodes and the reference (ground) node.
- One of the nodes of the circuits is chosen to be a reference node and node voltages are determined compared to this node.
- The number of node equations (size of the matrix) is the number of nodes in the circuit minus one.
- The choice of a reference node affects the form of the matrix.

Node voltage method - writing a matrix equation

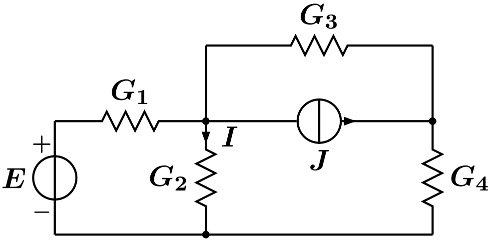
Matrix equation: $$\boxed{\boldsymbol{GU} = \boldsymbol{J}}$$

- Diagonal elements of $\boldsymbol{G}$ matrix are summation of conductances connected to each node.
- Other elements include conductances between the nodes with a negative sign
- In a source vector, the sign is plus if source brings current to node and minus if source takes current out of the node.

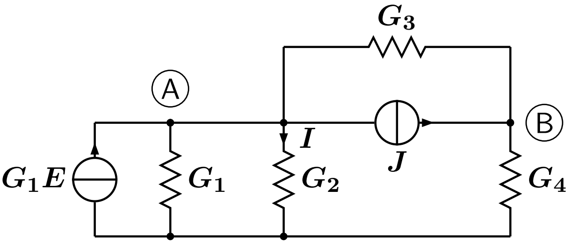
In node voltage method only current soruces are allowed so voltage sources should be transformed to current sources.

Example on node voltage method

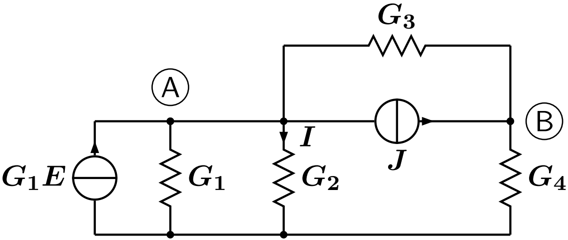
Calculate current I using node voltage method.



Only current sources are allowed $\rightarrow$ source transform



Example continues - writing equations



$$\begin{bmatrix} G_1 + G_2 + G_3 & -G_3 \\ -G_3 & G_3 + G_4 \end{bmatrix} \begin{bmatrix} U_{\mathrm{A}} \\ U_{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} G_1E - J \\ J \end{bmatrix}$$

Example continues - solving equations

$$\begin{bmatrix} G_1 + G_2 + G_3 & -G_3 \\ -G_3 & G_3 + G_4 \end{bmatrix} \begin{bmatrix} U_\mathrm{A} \\ U_\mathrm{B} \end{bmatrix} = \begin{bmatrix} G_1 E - J \\ J \end{bmatrix}$$

We can solve U_A using Cramer's rule:

$$U_A = \frac{\begin{vmatrix} G_1 E - J & -G_3 \\ J & G_3 + G_4 \end{vmatrix}}{\begin{vmatrix} G_1 + G_2 + G_3 & -G_3 \\ -G_3 & G_3 + G_4 \end{vmatrix}} = \frac{(G_1 E - J)(G_3 + G_4) + G_3 J}{(G_1 + G_2 + G_3)(G_3 + G_4) - G_3^2}$$

$$I = G_2 U_\mathrm{A}$$

Summary

Mesh current method

- unknowns = mesh currents
- only voltage sources
- $\boldsymbol{RI} = \boldsymbol{E}$

Node voltage method

- unknowns = node voltages
- only current sources
- $\boldsymbol{GU} = \boldsymbol{J}$

Dependent sources

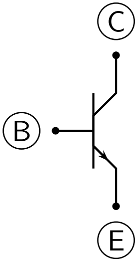
- Value of the independent source doesn't depend on other voltages or currents
- Value of dependent source depends on current or voltage in the circuit
- Dependent sources are used to model other components (for example transistors).

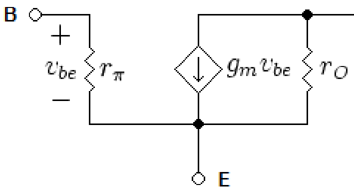
Four type of dependent sources:

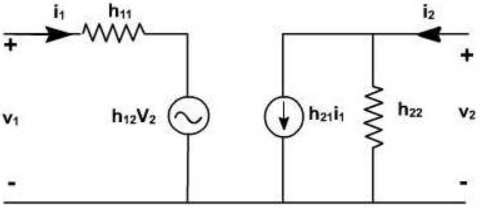
- CCVS - Current-Controlled Voltage Source ($\boldsymbol{E = RI}$)
- VCVS - Voltage-Controlled Voltage Source ($\boldsymbol{E = \alpha U}$)
- CCCS - Current-Controlled Current Source ($\boldsymbol{J = \beta I}$)
- VCCS - Voltage-Controlled Current Source ($\boldsymbol{J = GU}$)

Examples on transistor models

Dependent sources are needed e.g. in equivalent circuits of transistors







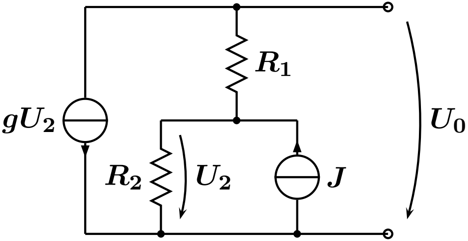
Don't be mixed although there are different graphic symbols in literature.

Dependent source in systematic methods

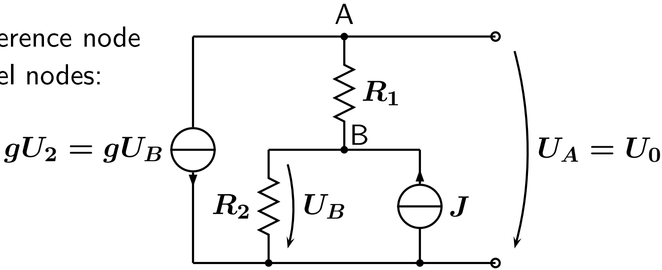
- The number of unknows cannot be larger than dimension of matrix in systematic methods $\Longrightarrow$ Value of dependent source has to be given using the mesh currents or node voltages.
- Dependent sources should move to the left-hand side of the equation before solving. $\boldsymbol{R}$ or $\boldsymbol{G}$ matrix become then asymmetric.
- Moving a term to the left-hand side is same thing when we move a term in an equation to other side of the equal sign. When we move the term
    1. sign changes
    2. same row (we move inside an equation)
    3. $\boldsymbol{k}$th column, if term is multiplying $\boldsymbol{k}$th unknown

Example on controlled source

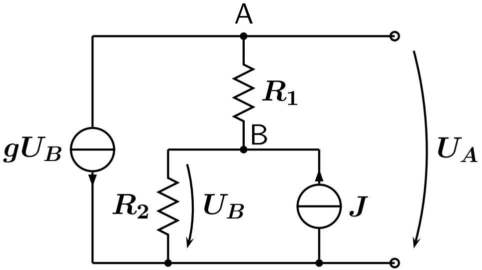
Use nodal analysis to find the voltage U_0.



Choose reference node and label nodes:



Example on controlled source continues



$$\begin{bmatrix} G_1 & -G_1 \\ -G_1 & G_1+G_2 \end{bmatrix} \begin{bmatrix} U_\mathrm{A} \\ U_\mathrm{B} \end{bmatrix} = \begin{bmatrix} -gU_\mathrm{B} \\ J \end{bmatrix}$$

Move unknowns to the left-hand side of the equation:

$$\begin{bmatrix} G_1 & -G_1+g \\ -G_1 & G_1+G_2 \end{bmatrix} \begin{bmatrix} U_\mathrm{A} \\ U_\mathrm{B} \end{bmatrix} = \begin{bmatrix} 0 \\ J \end{bmatrix}$$

Cramer's rule:

$$U_0 = U_\mathrm{A} = \frac{(G_1-g)J}{G_1(G_1+G_2)+G_1(-G_1+g)} = \frac{G_1-g}{G_1(G_2+g)}J$$