Galilein muunnos ja Newtonin 2.

$m\ddot{x}' \to$ mitä kun $x' = x - vt$
$t' = t$

$$\dot{x} = \frac{dx}{dt} = \underbrace{\frac{dt'}{dt}}_{1} \frac{d}{dt'}(x' + vt') = \frac{dx'}{dt'} - v$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d^2x'}{dt'^2}$$, koska $v =$ vakio

Jos voima riippuu vain suhteellisesta koordinaatista, $F(\bar{x} - \bar{R})$

niukkasen paikka

voiman lähteen paikka

$\Rightarrow F(\hat{x} - \hat{R}) = F(\bar{x}' - \bar{R}')$, koska "vt" termi on sama $\bar{x}'$:ssa ja $\bar{R}'$:ssa

$\Rightarrow$ on invariantti $\Rightarrow$ kiihtyvyys on sama kaikissa inertiaalikoordinaatistoissa.

Galilein muunnos: nopeuksien yhteenlasku esimerkki

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

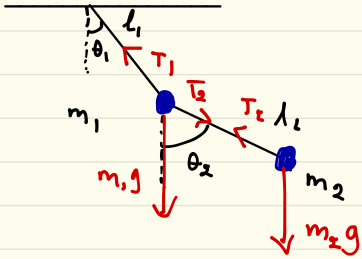
Tehdään 2 perättäistä muunnosta koordinaatistoihin, jotka liikkuvat nopeuksilla v_1 ja v_2.

$$\Rightarrow \begin{pmatrix} x'' \\ t'' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -v_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -v_1 \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 1 - v_2 \cdot 0 & -v_2 - v_1 \\ 0 \cdot 1 + 1 \cdot 0 & -v_1 \cdot 0 + 1 \cdot 1 \end{pmatrix}} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -v_1 - v_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

Eli Galilein muunnos missä nopeus $v_1 + v_2$!!

Double pendulum: equations of motion



$$\bar{r}_1 = l_1(\sin\theta_1, \cos\theta_1, 0)$$
$$\bar{r}_2 = \bar{r}_1 + l_2(\sin\theta_2, \cos\theta_2, 0)$$

$$\dot{\bar{r}}_1 = \bar{v}_1 = l_1\dot{\theta}_1(\cos\theta_1, -\sin\theta_1)$$
$$\ddot{\bar{r}}_1 = \bar{a}_1 = l_1\ddot{\theta}_1(\cos\theta_1, -\sin\theta_1) - l_1\dot{\theta}_1^2(\sin\theta_1, \cos\theta_1)$$
$$\dot{\bar{r}}_2 = \bar{v}_1 + l_2\dot{\theta}_2(\cos\theta_2, -\sin\theta_2)$$
$$\ddot{\bar{r}}_2 = \bar{a}_2 = \bar{a}_1 + l_2\ddot{\theta}_2(\cos\theta_2, -\sin\theta_2) - l_2\dot{\theta}_2^2(\sin\theta_2, \cos\theta_2)$$

Forces on 1: $\bar{F}_1 = T_1\frac{-\bar{r}_1}{|\bar{r}_1|} + T_2\frac{(\bar{r}_2-\bar{r}_1)}{|\bar{r}_2-\bar{r}_1|} + m_1\bar{g}$

$$= -\frac{T_1}{l_1}\bar{r}_1 + \frac{T_2}{l_2}(\bar{r}_2-\bar{r}_1) + m_1\bar{g}$$

Forces on 2: $-\frac{T_2}{l_2}(\bar{r}_2-\bar{r}_1) + m_2\bar{g}$

Eqns: $m_1\ddot{\bar{r}}_1 = \bar{F}_1$, $m_2\ddot{\bar{r}}_2 = \bar{F}_2$ $\Rightarrow$ 4 eqs, unknowns $\theta_1, \theta_2, T_1, T_2$ $\Rightarrow$ should work.

$$m_1 l_1(\ddot{\theta}_1\cos\theta_1 - \dot{\theta}_1^2\sin\theta_1) = -T_1\sin\theta_1 + T_2\sin\theta_2$$ x-comp for mass 1

$$-m_1 l_1(\ddot{\theta}_1\sin\theta_1 + \dot{\theta}_1^2\cos\theta_1) = -T_1\cos\theta_1 + T_2\cos\theta_1 + m_1 g$$ y-comp

$$m_2\left(l_1\ddot{\theta}_1\cos\theta_1 - l_1\dot{\theta}_1^2\sin\theta_1 + l_2\ddot{\theta}_2\cos\theta_2 - l_2\dot{\theta}_2^2\sin\theta_2\right) = -T_2\sin\theta_2$$

$$-m_2\left(l_1\ddot{\theta}_1\sin\theta_1 + l_1\dot{\theta}_1^2\cos\theta_1 + l_2\ddot{\theta}_2\sin\theta_2 + l_2\dot{\theta}_2^2\cos\theta_2\right) = -T_2\cos\theta_2 + m_2 g$$

use $\sin^2\theta + \cos^2\theta = 1$ and $\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1 = \sin(\theta_2-\theta_1)$

...

$$\begin{cases} l_1\ddot{\theta}_1 = (T_2/m_1)\sin(\theta_2-\theta_1) - g\sin\theta_1 & (1) \\ l_1\dot{\theta}_1^2 = (T_1/m_1) - (T_2/m_1)\cos(\theta_2-\theta_1) - g\cos\theta_1 & (2) \\ l_2\ddot{\theta}_2 = -(T_1/m_1)\sin(\theta_2-\theta_1) & (3) \\ l_2\dot{\theta}_2^2 = (T_2/m_2) + (T_2/m_1) - (T_1/m_1)\cos(\theta_2-\theta_1) & (4) \end{cases}$$

use (1) & (3) to express $T_1 = -m_1\dfrac{l_2\ddot{\theta}_2}{\sin(\theta_2-\theta_1)}$

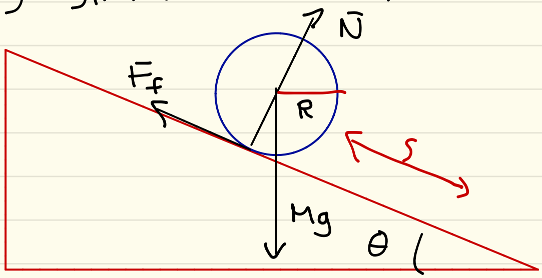
$$T_2 = m_1\frac{l_1\ddot{\theta}_1 + g\sin\theta_1}{\sin(\theta_2-\theta_1)}$$

$$\boxed{\begin{aligned} &\Rightarrow l_1\dot{\theta}_1^2 = \frac{-l_2\ddot{\theta}_2}{\sin(\theta_2-\theta_1)} - \frac{l_1\ddot{\theta}_1 + g\sin\theta_1}{\sin(\theta_2-\theta_1)}\cos(\theta_2-\theta_1) - g\cos\theta_1 \\ &-l_1\dot{\theta}_1^2\sin(\theta_2-\theta_1) = l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1\cos(\theta_2-\theta_1) + g\sin\theta_2 \end{aligned}}$$

This you could feed to a computer, set initial conditions $\theta_1(0)$, $\dot{\theta}_1(0)$, $\theta_2(0)$ and $\dot{\theta}_2(0)$ $\Rightarrow$ solve.

Rolling cylinder:

Typically we assume only center of mass physics and ignore the rest. As an example, let us discuss rolling cylinder on an incline



Rotating cylinder has kinetic energy

$E_R = \frac{1}{2} I \omega^2$, I = moment of inertia $= \frac{1}{2} M R^2$

ω = angular velocity

Method 1: using conservation of energy

$\frac{1}{2} M v^2 + \frac{1}{2} I \omega^2 - Mgh = 0$ h = how much dropped

no sliding just rolling $\Rightarrow v = \omega R$

$\Rightarrow v = \sqrt{\frac{4}{3} gh}$ which is less than $\sqrt{2gh}$ for center of mass only.

Average velocity $v_{avg} = v/2 = S/\Delta t \Rightarrow \Delta t = \frac{2S}{v}$

acceleration $a = v/\Delta t = \frac{v^2}{2S} = \frac{2g}{3} \sin\theta$, $\sin\theta = \frac{h}{S}$

Method 2: torque

If cylinder rolls angular momentum came from somewhere. Only friction F_f can cause torque others go through center of mass.

What is F_f? If ω is somehow too large so that static friction no longer works, cylinder slides. But here we could assume this doesn't happen.

along the incline: $Mg\sin\theta - F_f = Ma$ $(*)$

torque $\tau = F_f R = I\alpha$ (α = angular acceleration)

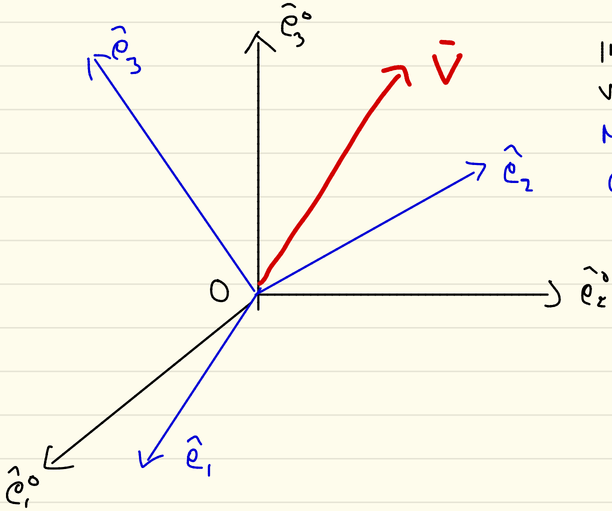
no slipping $\Rightarrow \alpha = a/R$

$\Rightarrow \frac{1}{2}MR^2\frac{a}{R} = F_f R \Rightarrow F_f = \frac{1}{2}Ma$

$(*) \Rightarrow Mg\sin\theta = \frac{3}{2}Ma \Rightarrow \boxed{a = \frac{2}{3}g\sin\theta}$

2. Accelerated coordinate systems:

For example: motion as observed from a lab fixed on rotating earth.



Inertial frame $\hat{e}_i^0$ with origin O $(i=\{1,2,3\})$

Moving orthonormal axes $\hat{e}_i$, same origin.

Some vector $\bar{V}$.

$$\bar{V} = \sum_{i=1}^{3} V_i^0 \hat{e}_i^0 \quad \text{OR} \quad \bar{V} = \sum_{i=1}^{3} V_i \hat{e}_i$$

In inertial frame: $\left(\frac{d\bar{V}}{dt}\right)_{\text{inertial}} = \sum_{i=1}^{3} \frac{dV_i^0}{dt} \hat{e}_i^0$

or $\left(\frac{d\bar{V}}{dt}\right)_{\text{inertial}} = \sum_i \frac{dV_i}{dt} \hat{e}_i + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$, where first term

is the rate of change in the body-fixed frame

$$\left(\frac{d\bar{V}}{dt}\right)_{\text{body}} = \sum_{i=1}^{3} \frac{dV_i}{dt} \hat{e}_i \Rightarrow \left(\frac{d\bar{V}}{dt}\right)_{\text{inertial}} = \left(\frac{d\bar{V}}{dt}\right)_{\text{body}} + \sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt}$$

3. Infinitesimal rotations

Consider small dt: $\hat{e}_i(t+dt) = \hat{e}_i(t) + d\hat{e}_i$

Orthonormal basis: $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$ (δ_{ij} = Kronecker delta) (*)

$\Rightarrow$ to lowest order in dt: $\hat{e}_i \cdot d\hat{e}_i = 0$ (**)

and $d\hat{e}_i$ is perpendicular to $\hat{e}_i$ (to lowest order in dt)

Can be expanded so: $d\hat{e}_i = \sum_{j=1}^{3} d\Omega_{ij}\,\hat{e}_j$ with some coeff. $d\Omega_{ij}$

(**) $\Rightarrow$ $d\Omega_{ii} = 0$ (& $d\hat{e}_i \cdot \hat{e}_j = d\Omega_{ij}$)

(*) $\Rightarrow$ $d\hat{e}_i \cdot \hat{e}_j + \hat{e}_i \cdot d\hat{e}_j = 0 \Rightarrow d\Omega_{ij} = -d\Omega_{ji}$

and we only have 3 independent elements

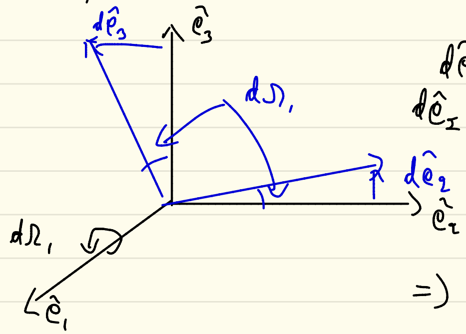
$$d\Omega_{12} \equiv d\Omega_3, \quad d\Omega_{23} \equiv d\Omega_1, \quad d\Omega_{31} \equiv d\Omega_2$$

So that for example $d\hat{e}_1 = d\Omega_3 \hat{e}_2 - d\Omega_2 \hat{e}_3 = d\bar{\Omega} \times \hat{e}_1$
where $d\bar{\Omega} \equiv d\Omega_1 \hat{e}_1 + d\Omega_2 \hat{e}_2 + d\Omega_3 \hat{e}_3$

... more generally $d\hat{e}_i = d\bar{\Omega} \times \hat{e}_i$

$d\bar{\Omega}$ interpretation: Combination of 3 separate rotations around each axes.

(4.) Right hand convention: $d\bar{\Omega}_1 = d\Omega_1 \hat{e}_1$



$$d\hat{e}_1 = 0$$
$$d\hat{e}_2 = d\Omega_1 \hat{e}_3 = d\bar{\Omega}_1 \times \hat{e}_2$$
$$d\hat{e}_3 = -d\Omega_1 \hat{e}_2 = d\bar{\Omega}_1 \times \hat{e}_3$$

$\Rightarrow$ formula OK.

On the other hand $d\hat{e}_i = \frac{d\hat{e}_i}{dt} dt = \frac{d\bar{\Omega}}{dt} \times \hat{e}_i \, dt = \bar{\omega} \times \hat{e}_i \, dt$

$\bar{\omega}$ = Instantaneous angular velocity vector of the rotating frame as seen in the inertial frame $\bar{\omega} = \frac{d\bar{\Omega}}{dt}$

Then $\sum_{i=1}^{3} V_i \frac{d\hat{e}_i}{dt} = \sum_{i=1}^{3} V_i \bar{\omega} \times \hat{e}_i = \bar{\omega} \times \bar{V}$ since $\bar{V} = \sum_{i=1}^{3} V_i \hat{e}_i$

$$\Rightarrow \boxed{\left(\frac{d\bar{V}}{dt}\right)_{inertial} = \left(\frac{d\bar{V}}{dt}\right)_{body} + \bar{\omega} \times \bar{V}}$$

Applies to any vector $\bar{V}$.

Example: $\bar{V} = \bar{r}$, $d\bar{\Omega} = d\phi \, \hat{z}$ (rotation around $\hat{z}$, $\bar{r}$ = position vector fixed in moving frame)

$$d\bar{r} = d\bar{\Omega} \times \bar{r}$$

2 consecutive rotations $d\bar{\Omega}_1$ & $d\bar{\Omega}_2 \Rightarrow \bar{r}_2 = \bar{r}_1 + d\bar{\Omega}_2 \times \bar{r}_1$

$$= \bar{r} + d\bar{\Omega}_1 \times \bar{r} + d\bar{\Omega}_2 \times (\bar{r} + d\bar{\Omega}_1 \times \bar{r}) \approx \bar{r} + (d\bar{\Omega}_1 + d\bar{\Omega}_2) \times \bar{r} + O(d\Omega^2)$$

Note: infinitesimal rotations commute (order doesn't matter)

However, finite rotatations do not commute

$$\Rightarrow \left(\frac{d\bar{r}}{dt}\right)_{inertial} = \frac{d\bar{\Omega}}{dt} \times \bar{r} = \bar{\omega} \times \bar{r} \quad \left(\left(\frac{d\bar{r}}{dt}\right)_{body} = 0 \text{ for } \bar{r}\right)$$

5. $d\bar{r}$ = vector $\Rightarrow$ so is $\bar{\omega}$ $\Rightarrow \left(\frac{d\bar{\omega}}{dt}\right)_{\text{inertial}} = \left(\frac{d\bar{\omega}}{dt}\right)_{\text{body}}$

(since $\bar{\omega} \times \bar{\omega} = 0$)

observers in inertial and rotating frames agree on the rate of change of $\bar{\omega}$.

Now take a coordinate of a moving particle $\bar{r}$

$$\Rightarrow \left(\frac{d\bar{r}}{dt}\right)_{\text{inertial}} = \left(\frac{d\bar{r}}{dt}\right)_{\text{body}} + \bar{\omega} \times \bar{r}$$

For dynamics we need accelerations which are vectors as well.

" $\left(\frac{d}{dt}\right)_{\text{inertial}} = \left(\frac{d}{dt}\right)_{\text{body}} + \omega \times$ "

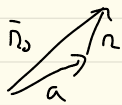
twice $\Rightarrow$

$$\left(\frac{d^2 \bar{r}}{dt^2}\right)_{\text{inertial}} = \left[\left(\frac{d}{dt}\right)_{\text{body}} + \omega \times\right]\left[\left(\frac{d\bar{r}}{dt}\right)_{\text{body}} + \bar{\omega} \times \bar{r}\right]$$

$$= \left(\frac{d^2\bar{r}}{dt^2}\right)_{\text{body}} + 2\bar{\omega} \times \left(\frac{d\bar{r}}{dt}\right)_{\text{body}} + \frac{d\bar{\omega}}{dt} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})$$

since $\frac{d\bar{\omega}}{dt}$ was independent of the frame.

$\Rightarrow$ Acceleration in rotating frame different from inertial frame



Add also translations? $\bar{r}_0 = \bar{r} + \bar{a}$

$$\Rightarrow \left(\frac{d^2 \bar{r}_0}{dt^2}\right)_{\text{inertial}} = \left(\frac{d^2 a}{dt^2}\right)_{\text{inertial}} + \left(\frac{d^2\bar{r}}{dt^2}\right)_{\text{inertial}}$$

$$= \left(\frac{d^2 a}{dt^2}\right)_{\text{inertial}} + \left(\frac{d^2\bar{r}}{dt^2}\right)_{\text{body}} + 2\bar{\omega} \times \left(\frac{d\bar{r}}{dt}\right)_{\text{body}} + \frac{d\bar{\omega}}{dt} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})$$

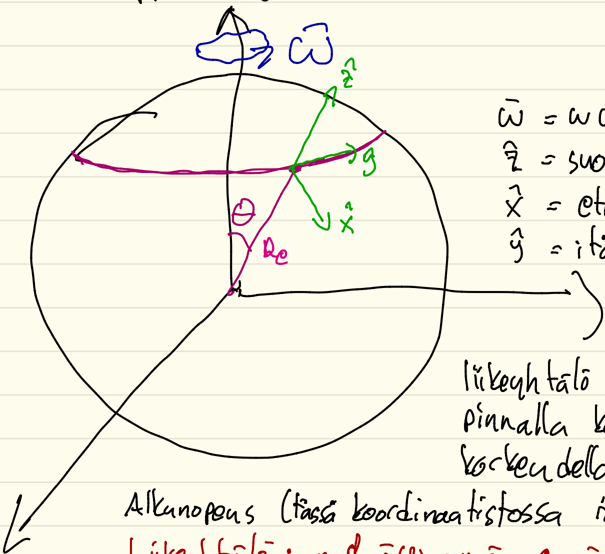
6. Newton's laws in accelerated system.

$$m\left(\frac{d^2\bar{r}_s}{dt^2}\right)_{inertial} = \bar{F}_e$$

$$\Rightarrow m\left(\frac{d^2\bar{r}}{dt^2}\right)_{body} = \bar{F}_e - m\left(\frac{d^2\bar{a}}{dt^2}\right)_{inertial} \underbrace{- 2m\bar{\omega}\times\left(\frac{d\bar{r}}{dt}\right)_{body}}_{\text{Coriolis}}$$

$$\underbrace{- m\bar{\omega}\times(\bar{\omega}\times\bar{r})}_{\text{Centrifugal}} \underbrace{- m\frac{d\bar{\omega}}{dt}\times\bar{r}}_{\text{Euler Force}}$$

Putoava kappale esimerkki:



$\bar{\omega} = \omega\cos\theta\,\hat{z} - \omega\sin\theta\,\hat{x}$
$\hat{z}$ = suoraan ylös
$\hat{x}$ = etelään
$\hat{y}$ = itään

liikeyhtälö koordinaatistossa maan pinnalla kappaleen alla. Kappale alussa korkeudella h eli $\bar{r}(0) = (0,0,h) = h\hat{z}$

Alkunopeus (tässä koordinaatistossa $\dot{r}(0) = 0$

Liikeyhtälö: $m\frac{d}{dt^2}\bar{r}(t) = m\bar{g} - 2m\bar{\omega}\times\bar{v}$ ← Coriolis-voima

$\bar{g} = -g_0\hat{z}$

Haetaan ratkaisua suoraan putoavan ratkaisun läheltä eli

$\bar{r}(t) = \bar{r}_0(t) + \bar{r}_1(t)$, missä $\bar{r}_0(t) = \left(-\frac{1}{2}g_0t^2 + h\right)\hat{z}$, $\bar{v}(t) = -g_0t\,\hat{z} + \underbrace{\frac{d}{dt}\bar{r}_1(t)}_{\bar{v}_1(t)}$

Sijoitus liikeyhtälöön:

$$m\left(-g_0\hat{z} + m\frac{d^2}{dt^2}\bar{r}_1(t)\right) = -mg_0\hat{z} - 2m\bar{\omega}\times\left(-g_0t\,\hat{z} + \bar{v}_1(t)\right)$$

$= m\frac{d^2}{dt^2}\bar{r}_1(t) \approx 2mg_0t\;\bar{\omega}\times\hat{z}$, koska $\bar{\omega}\times\bar{v}_1 \sim$ pieni × pieni ~ tosi pieni

$$\bar{\omega}\times\hat{z} = \omega\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ -\sin\theta & 0 & \cos\theta\\ 0 & 0 & 1\end{vmatrix} = +\sin\theta\,\hat{y}$$

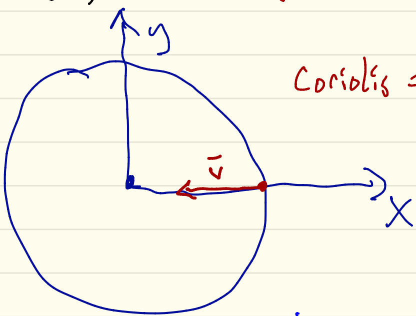
$\Rightarrow \bar{r}_1(t) = +\frac{1}{3}mg_0t^3\sin\theta\,\hat{y}$ ⇒ poikkeaa itään kun $\theta < \pi/2$. Miksi?

$h = 100\,m \Rightarrow t = \sqrt{2h/g_0} \approx 4.5\,s$

Poikkeama itään $\underline{\underline{2.2\,cm}}$

Mihin suuntaan ilma kiertää matalapaineen ympäri

- Pohjoisessa maan pyöriminen suunnilleen $\hat{z}$ suuntaan
- Ilmaan vaikuttaa painegradientti matalapaineen suuntaan (pyrkii täyttämään reiän)
- Coriolis $-2m\bar{\omega}\times\bar{v}$



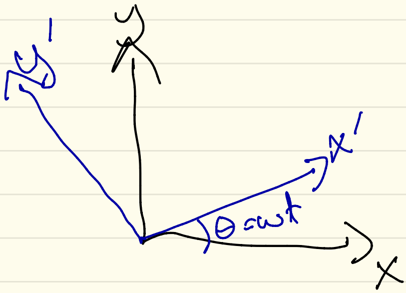
$$\text{Coriolis} = -2m\omega \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ -v & 0 & 0 \end{vmatrix}$$

$$= +2m\omega v\,\hat{y}$$

$\Rightarrow$ LÄHTEE POIKKEAMAAN Y-SUUNTAAN

$\Rightarrow$ VASTAPÄIVÄÄN POHJOISESSA

Koordinaattimuunnos 2D:ssä pyörivään koordinaatistoon?



$$\begin{cases} x' = x\cos\theta + y\sin\theta \\ y' = y\cos\theta - x\sin\theta \end{cases} \qquad (*)$$

Inertiaalikoordinaatistossa: $m\ddot{x} = F_x$, $m\ddot{y} = F_y$, $\bar{F} = (F_x, F_y)$

mutta koska nyt $\theta = \omega t$, yhtälö x':n ja y':n avulla on varmaan erilainen.

Käännetään $(*)$

$$\Rightarrow \quad x'\cos\theta = x\cos^2\theta + y\sin\theta$$
$$y'\sin\theta = y\sin\theta\cos\theta - x\sin^2\theta$$

$$\Rightarrow \quad x = x'\cos\theta - y'\sin\theta$$, koska $\sin^2\theta + \cos^2\theta = 1$
$$y = y'\cos\theta + x'\sin\theta$$

Nyt voidaan laskea aikaderivaatat. Huomaa $\dot{\theta} = \frac{d\theta}{dt} = \omega$

$$\dot{x} = \dot{x}'\cos\theta - \dot{y}'\sin\theta - x'\omega\sin\theta - y'\omega\cos\theta$$

$$\ddot{x} = \ddot{x}'\cos\theta - \dot{x}'\omega\sin\theta - \ddot{y}'\sin\theta - \dot{y}'\omega\cos\theta - \dot{x}'\omega\sin\theta - x'\omega^2\cos\theta - \dot{y}'\omega\cos\theta + y'\omega^2\sin\theta$$

$$(1) \quad = \ddot{x}'\cos\theta - \ddot{y}'\sin\theta - 2\dot{x}'\omega\sin\theta - 2\dot{y}'\omega\cos\theta - x'\omega^2\cos\theta + y'\omega^2\sin\theta$$

$\dot{y} = \dot{y}'\cos\theta - y'\omega\sin\theta + \dot{x}'\sin\theta + x'\omega\cos\theta$

$$\ddot{y} = \ddot{y}'\cos\theta + \ddot{x}'\sin\theta - 2\dot{y}'\omega\sin\theta + 2\dot{x}'\omega\cos\theta \qquad (2)$$
$$- y'\omega^2\cos\theta + x'\omega^2\sin\theta$$

$(1)\cdot\cos\theta + (2)\cdot\sin\theta = \ddot{x}' + x'\omega^2 - 2\dot{y}'\omega$

$(1)\cdot(-\sin\theta) + (2)\cdot\cos\theta = \ddot{y}' + y'\omega^2 + 2\dot{x}'\omega$

$\bar{v}' = (\dot{x}', \dot{y}')$, $\bar{\omega} = \omega\hat{e}_z$

$\Rightarrow m\ddot{x}' = -m\omega^2 x' + 2m\omega\dot{y}' + F_x'$ voiman komponentti x' suuntaan

$m\ddot{y}' = -m\omega^2 y' - 2m\omega\dot{x}' + F_y'$

$$-\bar{\omega}\times\bar{v}' = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 0 & 0 & -\omega \\ v_x' & v_y' & 0 \end{vmatrix} = \omega v_y'\hat{e}_x - \omega v_x'\hat{e}_y$$

Keskipakoisvoima $-m\omega^2\bar{r}$ ilmestyi
Samoin Coriolis-voima $-2m\bar{\omega}\times\bar{v}$