Quantizations and time-frequency transforms

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Remark: These slides are "nice to know" extra information for the course in Fourier Analysis. There will be a course on **Time-Frequency Analysis** in the academic year 2018–2019.

Abstract

A Cohen class time-frequency transform of signals $u, v : \mathbb{R} \to \mathbb{C}$ is a time-frequency invariant sesquilinear mapping $(u, v) \mapsto C(u, v)$, where the time-frequency distribution

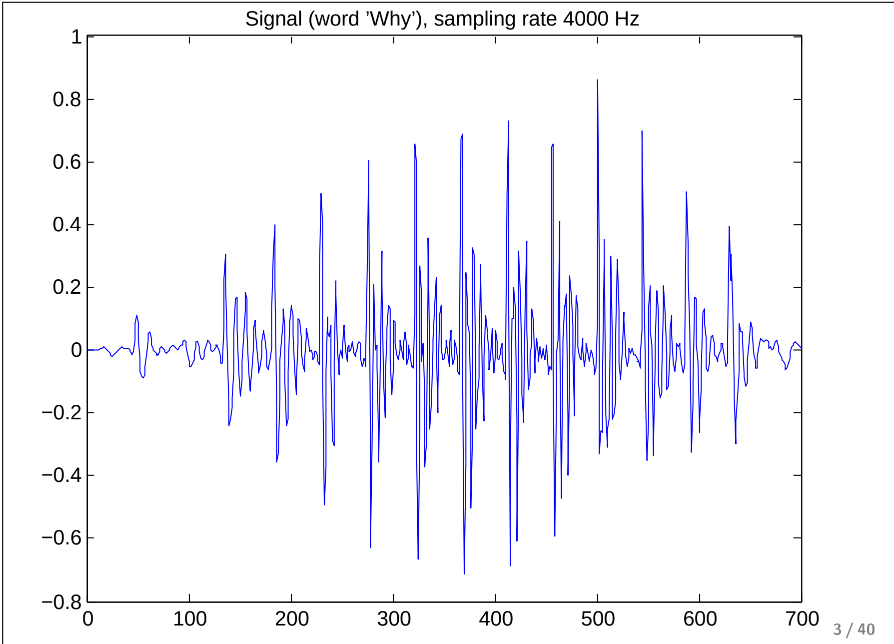
$$C[u] = C(u, u) : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{R}$$

can be thought as a phase-space energy density of u. For instance, all the spectrograms are such energy densities. We study properties of different time-frequency transforms C and their related pseudo-differential operator quantizations $a \mapsto a_C$ defined by the Hilbert space duality

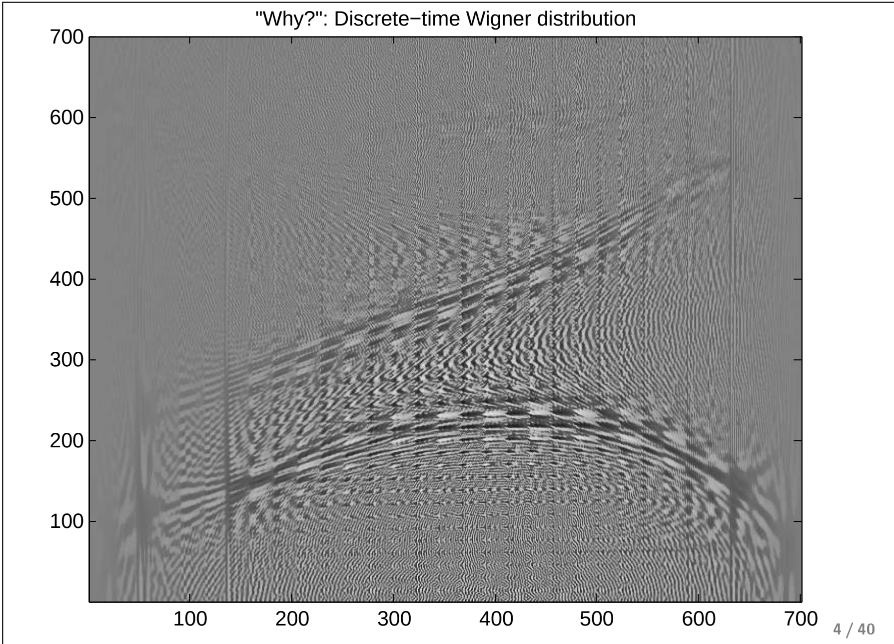
$$\langle u, a_C v \rangle_{L^2(\mathbb{R})} = \langle C(u, v), a \rangle_{L^2(\mathbb{R} \times \widehat{\mathbb{R}})}.$$

We also present computed examples from acoustic signal processing, quantum mechanics and medical sciences. When and how often something happens in signals? By properly quantizing these questions, we obtain the Born–Jordan transform.

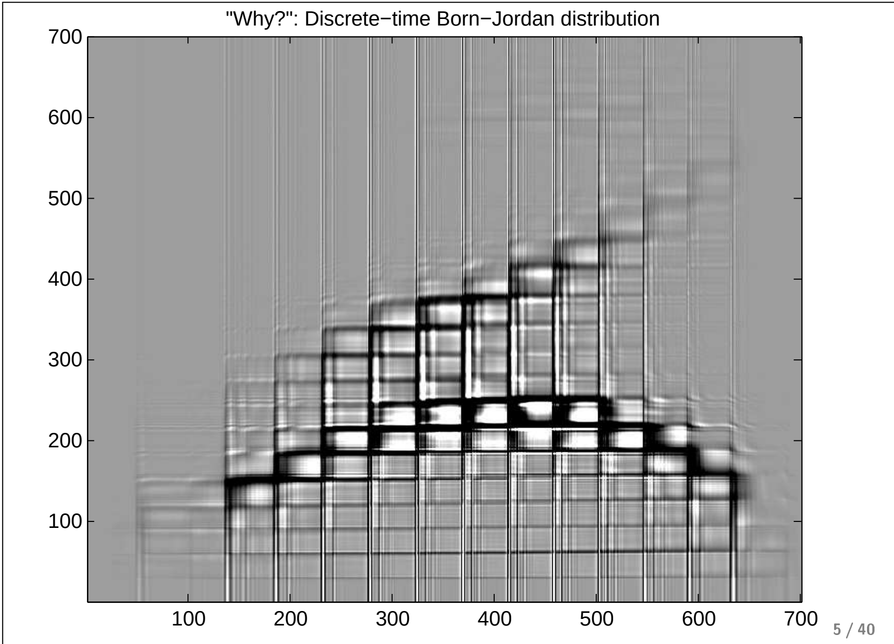
Waveform of speech...



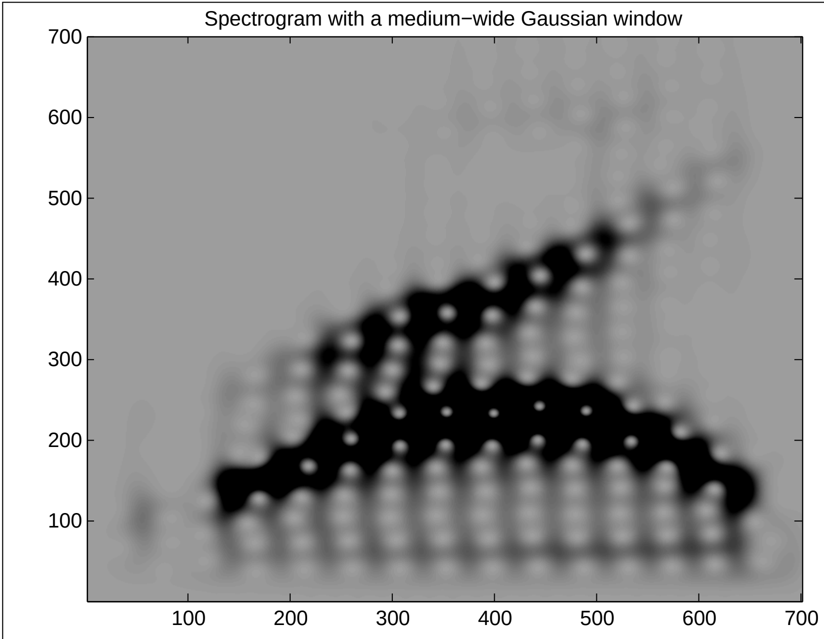
Wigner for speech (compare to next Born–Jordan...)



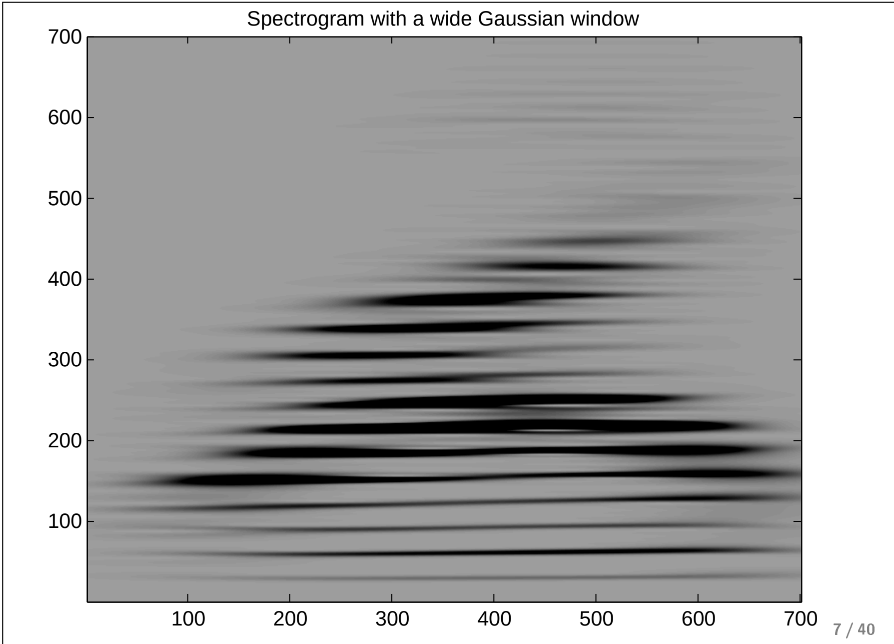
… Born–Jordan for speech (compare to next spectrogram…)



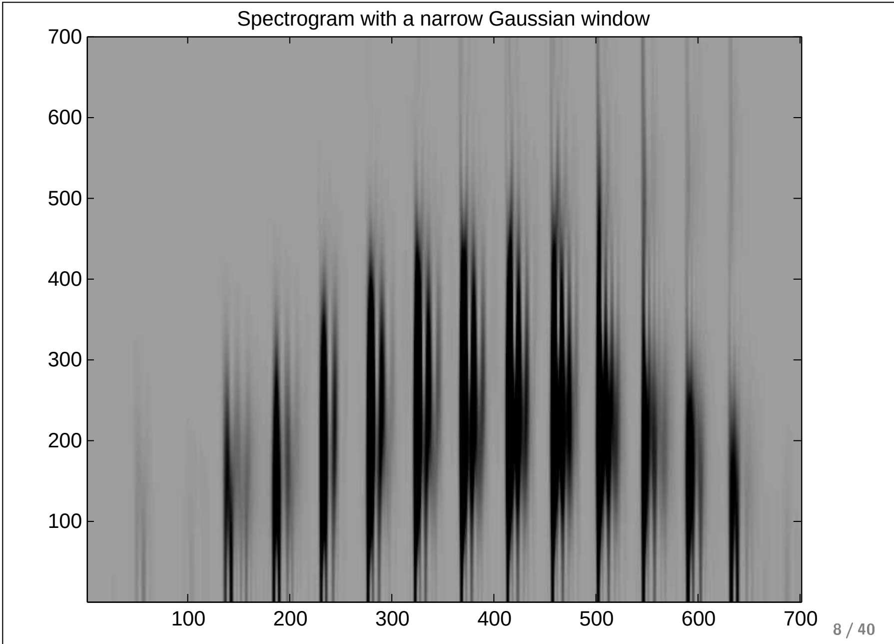
… a spectrogram (compare to previous Born–Jordan)



… a spectrogram with wide Gaussian window…



… a spectrogram with narrow Gaussian window



Some history

- Tools: **Fourier analysis, FFT**!
- 1920s foundations of quantum mechanics (Heisenberg, Born, Jordan, Schrödinger; Dirac, Wigner, Weyl, von Neumann)
- Spectrograms (early 1940s Bell Labs, 1944–1946 Gabor)
- 1966 Leon Cohen's class of time-frequency distributions (physics, signal processing)
- 1960s pseudodifferential operators (Hörmander et al)
- 2010 Born-Jordan L^2-continuity (Boggiatto–De Donno–Oliaro)
- 2011 Born-Jordan uncertainty (Boggiatto–Oliaro–Carypis)

Notations

Time-like variables: Latin letters $t, x, y \in \mathbb{R}$.
Frequency-like variables: respective Greek letters $\tau, \xi, \eta \in \widehat{\mathbb{R}} \cong \mathbb{R}$.
Time-frequency plane

$$\mathbb{R} \times \widehat{\mathbb{R}} = \left\{ (x, \eta) : \; x \in \mathbb{R}, \; \eta \in \widehat{\mathbb{R}} \right\}.$$

Signal $u : \mathbb{R} \to \mathbb{C}$ has Fourier transform $\mathscr{F} u = \widehat{u} : \widehat{\mathbb{R}} \to \mathbb{C}$,

$$\widehat{u}(\eta) := \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi y \cdot \eta} \, u(y) \, \mathrm{d}y.$$

Inner product $\langle u, v \rangle \in \mathbb{C}$ of signals $u, v : \mathbb{R} \to \mathbb{C}$,

$$\langle u, v \rangle := \int u(x) \, v(x)^* \, \mathrm{d}x, \tag{1}$$

where $v(x)^* = \overline{v(x)}$ the complex conjugate. *Energy* of signal u:

$$\|u\|^2 = \langle u, u \rangle = \int |u(x)|^2 \, \mathrm{d}x \quad \geq \quad 0. \tag{2}$$

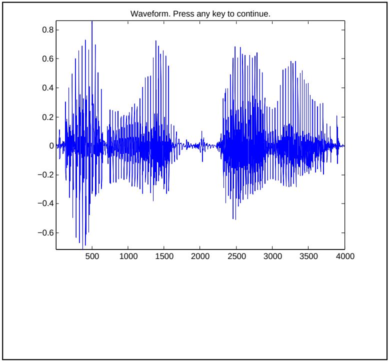
Fourier preserves inner products (and energy), $\langle \widehat{u}, \widehat{v} \rangle = \langle u, v \rangle$.

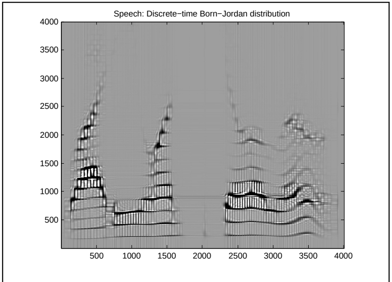
Idea of time-frequency analysis

Signals $u, v : \mathbb{R} \to \mathbb{C}$
of finite energy: $u, v \in \mathscr{H} = L^2(\mathbb{R})$.

Time-frequency transform
$$C(u,v) : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C},$$
t.-f. distribution ("energy density")
$$C(u,u) = C[u] : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{R},$$
where the signal equivalence class
$[u] = \{\lambda u : \ \lambda \in \mathbb{C}, \ |\lambda| = 1\}$.

Time-frequency weight (symbol)
$$a : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C},$$
C-quantization $a \mapsto a_C = a_C(X, D)$:
$\langle u, a_C v\rangle_{L^2(\mathbb{R})} = \langle C(u,v), a\rangle_{L^2(\mathbb{R}\times\widehat{\mathbb{R}})}$.





Cohen class time-frequency transforms C

Signals $u, v : \mathbb{R} \to \mathbb{C}$ have *time-frequency transform*

$$C(u, v) : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C},$$

and $C[u] := C(u, u) : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$ is *time-frequency distribution* (or "**energy density**").

[Cohen 1966, Gröchenig 2001, T. 2016] Basic requirements for C:

- $C(u, v)(0, 0) = \langle u, \delta_C v \rangle$ for a bounded operator δ_C on $\mathscr{H}$.
- For $v(x) = \mathrm{e}^{\mathrm{i}2\pi x \cdot \xi} u(x - y)$, time-frequency shift invariance

$$C[v](x, \eta) = C[u](x - y, \eta - \xi).$$

δ_C is the C-quantized pseudodifferential operator with symbol

$$\delta = \delta_{(0,0)}.$$

Then $C(u, v) = k_C * W(u, v)$ for a tempered distribution k_C, where $W(u, v)$ is the *Wigner transform*.
Normalization $\iint C[u](x, \eta)\,\mathrm{d}x\,\mathrm{d}\eta = \|u\|^2$: $\iint k_C(x, \eta)\,\mathrm{d}x\,\mathrm{d}\eta = 1$.

Potential extra conditions on $C(u, v)$?

- **Symmetry** $C(v, u) = C(u, v)^*$ (**real** energy density $C[u]$).
- Should $[u] \mapsto C[u] = C(u, u)$ be **invertible**?
- Should $[u] \mapsto C[u]$ be **robust** under noise?
- Should C have correct **marginal** energy densities? This means $\int C[u](x, \eta)\, \mathrm{d}\eta = |u(x)|^2$ and $\int C[u](x, \eta)\, \mathrm{d}x = |\widehat{u}(\eta)|^2$.
- Should C be **scale-invariant**? That is, if $v(x) := \sqrt{|\lambda|}u(\lambda x)$ for $0 \neq \lambda \in \mathbb{R}$ then $C[v](x, \eta) = C[u](\lambda x, \eta/\lambda)$.
- Should C be **time-local**? This means that if $u(x) = 0$ whenever $x \notin [a, b]$, then $C[u](x, \eta) = 0$ whenever $x \notin [a, b]$.
- Should C be **frequency-local**? This means that if $\widehat{u}(\eta) = 0$ whenever $\eta \notin [\alpha, \beta]$, then $C[u](x, \eta) = 0$ whenever $\eta \notin [\alpha, \beta]$.
- **Comb-to-grid** property: Since "ticking-of-a-clock" $u(x) := \sum_{k \in \mathbb{Z}} \delta_k(x) = \sum_{\kappa \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi x \cdot \kappa}$, should $C[u]$ show vertical and horizontal Dirac delta lines at integers $k, \kappa \in \mathbb{Z}$?

(The **Born–Jordan transform** $C = Q$ follows by requiring only scale-invariance, time-locality and the comb-to-grid property; and then Q satisfies all the other mentioned extras, too!)

Why spectrograms fail?

Spectrogram $\mathrm{Spec}_w[u] = C[u]$ for normalized window w is given by

$$C[u](x,\eta) := \left| \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta}\; u(y)\, w(y-x)^* \,\mathrm{d}y \right|^2 . \tag{3}$$

Here $C(u,v)(0,0) = \langle u, \delta_C v \rangle$ for the original localization

$$\delta_C v := \langle v, w \rangle w. \tag{4}$$

This is the orthogonal projection onto the 1-dimensional subspace spanned by w. Alternatively, for $\tilde{w}(t) := w(-t)$, here

$$C[u] = W[\tilde{w}] * W[u].$$

By unitary equivalence, we may assume $w(t) = 2^{1/4}\mathrm{e}^{-\pi t^2}$. But then $C[u]$ is just “melted down” version of $W[u]$ by heat equation. Hence, spectrograms always destroy information!

Order on $\mathbb{R}$ characterizes Born–Jordan [V.T.]

Studying a quantum particle on the real line $\mathbb{R}$, we ask:

(A) Is particle on the right?

(B) Is particle moving right?

So, we just ask for the **directions** of location and of movement
(Separate "right from left", "up from down", "future from past"...)
In other words, the **order** relation on $\mathbb{R}$ is essential here.

The uncertainty in (A, B) characterizes the Born–Jordan transform, leading to sharp time-frequency (phase-space) analysis.

Other time-frequency transforms **do not** properly answer to (A, B).

This has important consequences in signal processing
and quantum mechanics.

Direction of position

Wavefunction $\psi \in \mathscr{H} = L^2(\mathbb{R})$ describing a quantum particle on $\mathbb{R}$.
By Max Born, probability of finding "position" right of $x \in \mathbb{R}$ is

$$\int_x^\infty |\psi(y)|^2 \, \mathrm{d}y \quad \in \quad [0,1]. \tag{5}$$

Localization to $[x, \infty) \subset \mathbb{R}$ is given by projection $A_x : \mathscr{H} \to \mathscr{H}$,

$$A_x u(y) := \begin{cases} u(y) & \text{when } y \geq x, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Observable A_x: "**Is particle having position right of** x**?**"
"**Yes**" (eigenvalue 1) with probability $\int_x^\infty |\psi(y)|^2 \, \mathrm{d}y$,
and we find the updated wavefunction $u/\|u\|$ with $u = A_x\psi$.
"**No**" (eigenvalue 0) with probability $\int_{-\infty}^x |\psi(y)|^2 \, \mathrm{d}y$,
and the wavefunction becomes $v/\|v\|$ with $v = \psi - A_x\psi$.

Direction of momentum

Change from "position" x to "momentum" η by Fourier transform.
By Max Born, probability of finding "momentum" above $\eta \in \widehat{\mathbb{R}}$ is

$$\int_\eta^\infty |\widehat{\psi}(\xi)|^2 \,\mathrm{d}\xi \quad \in \quad [0,1]. \tag{7}$$

Localization to $[\eta, \infty) \subset \widehat{\mathbb{R}}$ is given by projection $B_\eta : \mathscr{H} \to \mathscr{H}$,

$$\widehat{B_\eta u}(\xi) := \begin{cases} \widehat{u}(\xi) & \text{when } \xi \geq \eta, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Observable B_η: "**Is particle having momentum above η?**"
"**Yes**" (eigenvalue 1) with probability $\int_\eta^\infty |\widehat{\psi}(\xi)|^2 \,\mathrm{d}\xi$,
and we find the updated wavefunction $u/\|u\|$ with $u = B_\eta \psi$.
"**No**" (eigenvalue 0) with probability $\int_{-\infty}^\eta |\widehat{\psi}(\xi)|^2 \,\mathrm{d}\xi$.
and the wavefunction becomes $v/\|v\|$ with $v = \psi - B_\eta \psi$.

Expectation of directional uncertainty

Uncertainty observable of the observable pair (A, B) is

$$-\mathrm{i}2\pi\,[A, B] = -\mathrm{i}2\pi\,(AB - BA)\,. \tag{9}$$

An application of the Cauchy–Schwarz inequality yields

$$\langle -\mathrm{i}2\pi\,[A, B]u, u\rangle = 4\pi\,\mathrm{Im}\langle Au, Bu\rangle \leq 4\pi\,\|Au\|\,\|Bu\|. \tag{10}$$

This gives the Heisenberg uncertainty inequality

$$|\langle -\mathrm{i}2\pi\,[A, B]\rangle| \leq 4\pi\,(\Delta A)\,(\Delta B), \tag{11}$$

where in state u observable M has *expectation* $\langle M\rangle := \langle Mu, u\rangle$ and *uncertainty* $\Delta M := \|Mu - \langle M\rangle u\|$. For $u, v \in \mathscr{H}$, define

$$Q(u, v)(x, \eta) := \langle -\mathrm{i}2\pi\,[A_x, B_\eta]u, v\rangle. \tag{12}$$

We call $Q(u, v) : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$ **the** time-frequency *transform*, and $Q[u] = Q(u, u) : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{R}$ **the** time-frequency *distribution*.
$Q[\psi](x, \eta)$ is the expectation of uncertainty of (A_x, B_η) in state ψ.
Especially: $Q[\psi](0, 0)$ is the
expectation of uncertainty in directional location and movement,

Born–Jordan transform

... and a brief calculation yields the **Born–Jordan transform**

$$Q(u,v)(x,\eta) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta} \, \frac{1}{y} \int_x^{x+y} u(t)\, v(t-y)^* \mathrm{d}t\, \mathrm{d}y. \qquad (13)$$

$Q[\psi] = Q(\psi,\psi)$ is a "quasi-probability distribution" of ψ, or $Q[u] = Q(u,u)$ is an "energy density" of u.

Alternatively, the Born–Jordan transform is given by

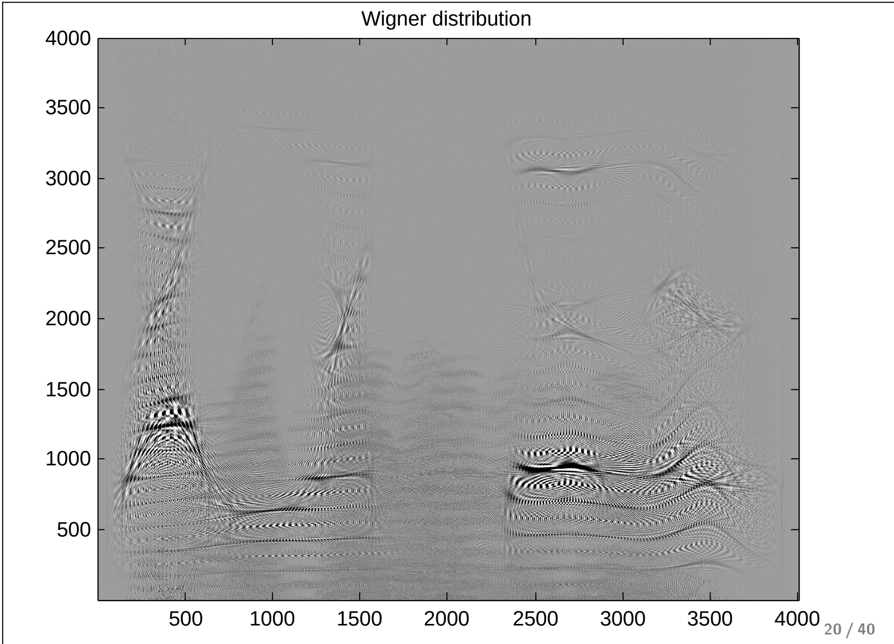
$$FQ(u,v)(\xi,y) = \operatorname{sinc}(\xi\cdot y)\ FW(u,v)(\xi,y), \qquad (14)$$

where $F = \mathscr{F} \otimes \mathscr{F}^{-1}$ is the symplectic Fourier transform, and the **Wigner transform** $W(u,v)$ is defined by

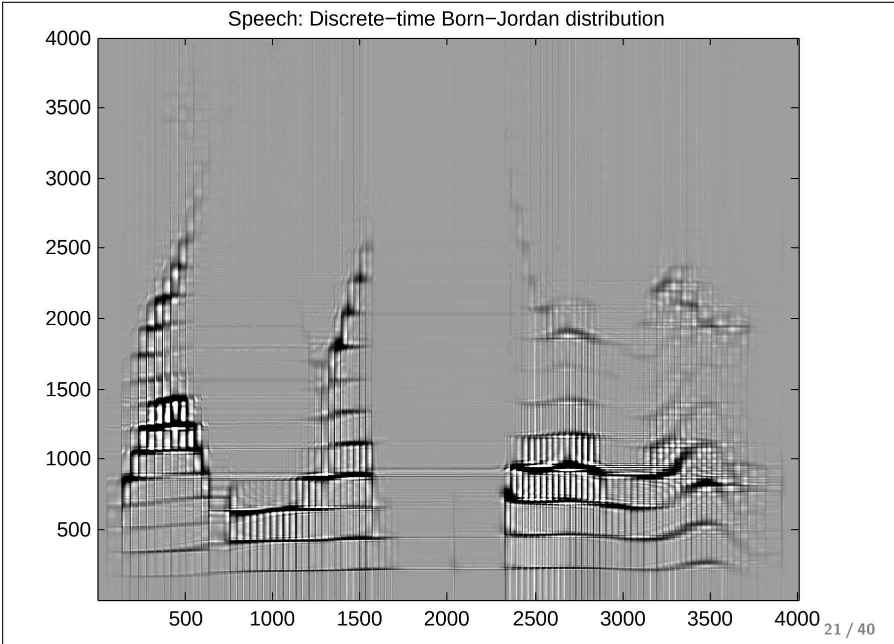
$$W(u,v)(x,\eta) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta}\, u(x+y/2)\, v(x-y/2)^*\, \mathrm{d}y. \qquad (15)$$

Unfortunately, Wigner has a very bad property:
it is **extremely sensitive** to noise (unlike the non-unitary Q. Also, Q is not causal nor positive.)

Wigner distribution is sensitive to noise: speech example



… and corresponding Born–Jordan distribution



Bound for Born–Jordan energy density [V.T. 2016]

There is the optimal Born–Jordan bound

$$|Q(u,v)(x,\eta)| \leq \pi \,\|u\|\,\|v\| \tag{16}$$

for all $(x,\eta) \in \mathbb{R} \times \widehat{\mathbb{R}}$. Especially, $|Q[u](x,\eta)| \leq \pi \,\|u\|^2$.

Proof.

"Point localization at the origin" $L = \delta_Q$:

$$\begin{aligned} \langle Lu, v\rangle &= Q(u,v)(0,0) \\ &= \int \frac{1}{z} \int_{-z/2}^{z/2} u(t+z/2)\, v(t-z/2)^* \,\mathrm{d}t\,\mathrm{d}z \\ &= \iint K_L(x,y)\, u(y)\, v(x)^* \,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

with the Schwartz kernel

$$K_L(x,y) = \begin{cases} |x-y|^{-1} & \text{if } xy < 0, \\ 0 & \text{if } xy \geq 0. \end{cases}$$

Proof of the bound for energy density

From

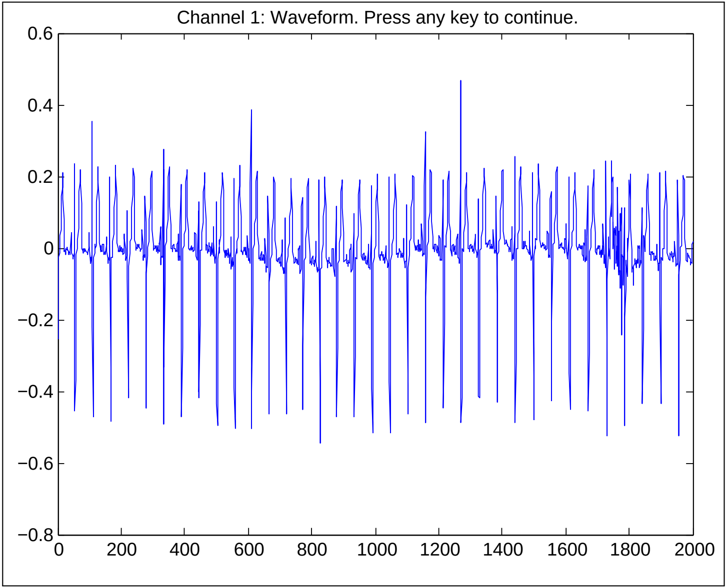
$$\begin{aligned}\langle Lu, v\rangle &= \int_{-\infty}^{0}\int_{0}^{\infty} \frac{u(y)\,v(x)^*}{y-x}\,\mathrm{d}y\,\mathrm{d}x + \int_{0}^{\infty}\int_{-\infty}^{0} \frac{u(y)\,v(x)^*}{x-y}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{0}^{\infty}\int_{0}^{\infty} \frac{u(y)v(-x)^*}{y+x}\,\mathrm{d}y\,\mathrm{d}x + \int_{0}^{\infty}\int_{0}^{\infty} \frac{u(-y)v(x)^*}{x+y}\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

denoting $u = Pu + Nu$, where $Pu(x) = \mathbf{1}_{\mathbb{R}^+}(x)\,u(x)$, we get

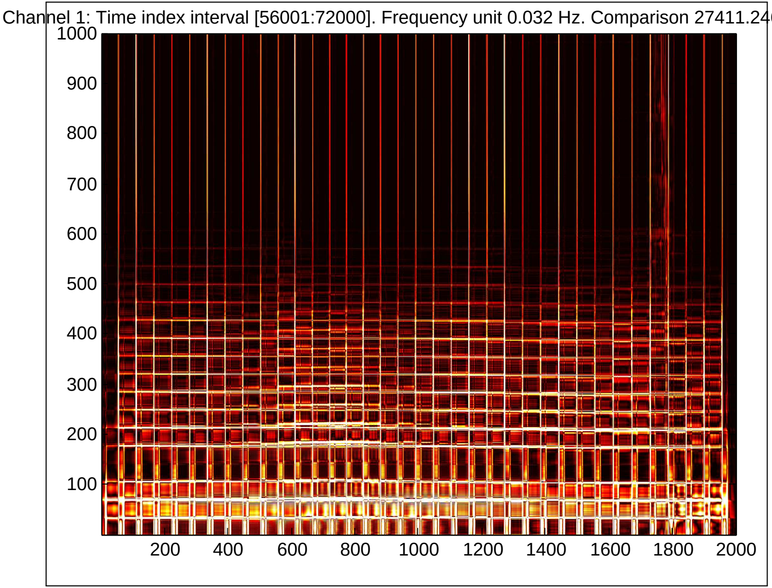
$$\begin{aligned}|\langle Lu, v\rangle| &\overset{Hilbert}{\leq} \pi\,\|Pu\|\,\|Nv\| + \pi\,\|Nu\|\,\|Pv\| \\ &= \pi\,(\|Pu\|, \|Nu\|)\cdot(\|Nv\|, \|Pv\|) \\ &\overset{Cauchy-Schwarz}{\leq} \pi\sqrt{\|Pu\|^2 + \|Nu\|^2}\sqrt{\|Nv\|^2 + \|Pv\|^2} \\ &= \pi\,\|u\|\,\|v\|.\end{aligned}$$

Especially, $|\langle Lu, u\rangle| \leq \pi\,\|u\|^2$. □

EKG (Electro-KardioGram)



EKG: Born–Jordan distribution (absolute value)



Another characterization for Born–Jordan [V.T. 2016]

Cohen class transforms: $C = k * W$ for some $k \in \mathscr{S}'(\mathbb{R} \times \widehat{\mathbb{R}})$.
Necessary and sufficient (i,ii,iii) for $C = Q$:

(i) C is **scale-invariant**. This means that if $v(x) = \lambda^{1/2} u(\lambda x)$ for $\lambda > 0$ then $C[v](x, \eta) = C[u](\lambda x, \eta/\lambda)$.

(ii) C is **time-local**. This means that if $u(x) = 0$ whenever $x \notin [a, b] \subset \mathbb{R}$ then $C[u](x, \eta) = 0$ whenever $x \notin [a, b] \subset \mathbb{R}$.

(iii) C maps **Dirac delta comb** to **Dirac delta grid**. This means

$$C[\delta_{\mathbb{Z}}](x, \eta) = \delta_{\mathbb{Z}}(x) + \delta_{\mathbb{Z}}(\eta) - 1,$$

where the Dirac delta comb is

$$\delta_{\mathbb{Z}}(x) = \sum_{k \in \mathbb{Z}} \delta_k(x) = \sum_{\kappa \in \mathbb{Z}} \mathrm{e}^{\mathrm{i}2\pi x \cdot \kappa},$$

with δ_k being the Dirac delta distribution at $k \in \mathbb{Z}$. Think $\delta_{\mathbb{Z}}$ as a ticking-of-a-clock. Notice that $\mathscr{F}(\delta_{\mathbb{Z}}) = \delta_{\mathbb{Z}}$.

Wigner distribution has properties (i,ii) but not (iii).
Spectrograms satisfy none of the properties (i,ii,iii).

Proof idea of Born–Jordan characterization

Let $FC[u] = \phi\, FW[u]$. We must show $\phi(\xi, y) = \mathrm{sinc}(\xi \cdot y)$.

(i) $\Rightarrow$ $\phi(\xi, y) = \widehat{\varphi}(\xi \cdot y)$ for some $\varphi \in \mathscr{S}'(\mathbb{R})$.

(ii,i) $\Rightarrow$ $\varphi(x) = 0$ for almost all $|x| > 1/2$.

So $\widehat{\varphi} : \mathbb{C} \to \mathbb{C}$ analytic by Paley–Wiener–Schwartz Thm.

(iii,ii,i) $\Rightarrow$ $\widehat{\varphi}(k) = \begin{cases} 1 & \text{if} \quad k = 0, \\ 0 & \text{if} \quad k \in \mathbb{Z} \setminus \{0\}. \end{cases}$

So $\widehat{u} = \widehat{\varphi}/\mathrm{sinc} : \mathbb{C} \to \mathbb{C}$ is analytic. Then $\varphi = u * \mathbf{1}_{[-1/2,1/2]}$ gives

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \varphi(x + k) &= \sum_{k \in \mathbb{Z}} \int_{[-1/2,1/2]} u(x + k - y)\, \mathrm{d}y \\ &= \int u(x)\, \mathrm{d}x = \widehat{u}(0) = \frac{\widehat{\varphi}(0)}{\mathrm{sinc}(0)} = 1, \end{aligned}$$

so $\varphi(x) = 1$ for almost all $x \in [-1/2, 1/2]$. Thus we obtain

$$\widehat{\varphi}(\xi \cdot y) = \int \mathrm{e}^{-\mathrm{i}2\pi x y \cdot \xi}\, \varphi(x)\, \mathrm{d}x = \int_{-1/2}^{1/2} \mathrm{e}^{-\mathrm{i}2\pi x y \cdot \xi}\, \mathrm{d}x = \mathrm{sinc}(\xi \cdot y). \qquad \square$$

Inversion of Born–Jordan distribution [V.T.]

Mapping $u \mapsto Q[u]$ is not invertible: $Q[\lambda u] = |\lambda|^2 Q[u]$.
But mapping $[u] \mapsto Q[u]$ is invertible, where the equivalence class

$$[u] = \{\lambda u : \ \lambda \in \mathbb{C}, \ |\lambda| = 1\}.$$

It is easy to see invertibility of Wigner $[u] \mapsto W[u]$, and by (14) also $[u] \mapsto Q[u]$ can be inverted. Notice the marginals:

$$\int_{\mathbb{R}} Q[u](x,\eta)\,\mathrm{d}\eta = |u(x)|^2, \quad \int_{\widehat{\mathbb{R}}} Q[u](x,\eta)\,\mathrm{d}x = |\widehat{u}(\eta)|^2. \tag{17}$$

Yet these marginals are not enough to find $[u]$.
Suppose $u(x) \neq 0$ for Schwartz function u. Then we have inversion

$$u(x+h) = \frac{h}{u(x)^*} \sum_{k=0}^{\infty} \partial_1 R(x - kh, h) \tag{18}$$

for all $h \neq 0$, where ∂_1 is the partial derivative in the first variable,

$$R(x,y) := \int_{\widehat{\mathbb{R}}} \mathrm{e}^{\mathrm{i}2\pi y\cdot\eta}\, Q[u](x,\eta)\,\mathrm{d}\eta. \tag{19}$$

Quantization $a \mapsto a_C$

Cohen class transform C gives the corresponding *quantization*

$$a \mapsto a_C, \quad a_C = a_C(X, D)$$

from *symbols* $a : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$ to *pseudo-differential operators* a_C.
Such linear operator $a_C = a_C(X, D)$ is defined by the duality

$$\langle u, a_C v \rangle = \langle C(u, v), a \rangle. \tag{20}$$

Properties of C are reflected in $a \mapsto a_C$. E.g. marginal conditions

$$\int_{\mathbb{R}} C[u](x, \eta)\, \mathrm{d}\eta = |u(x)|^2, \quad \int_{\mathbb{R}} C[u](x, \eta)\, \mathrm{d}x = |\widehat{u}(\eta)|^2, \tag{21}$$

mean that if $a(x, \eta) = f(x)$ and $b(x, \eta) = \widehat{g}(\eta)$ then
$a_C u(x) = f(x)\, u(x)$ (multiplication) and
$b_C u(x) = g * u(x) = \int_{\mathbb{R}} g(x - y)\, u(y)\, \mathrm{d}y$ (convolution).
Especially, if (21) and $a(x, \eta) = x$ and $b(x, \eta) = \eta$,
then $a_C = X$ and $b_C = D$, where

$$Xu(x) = x\, u(x), \quad Du(x) = \frac{1}{\mathrm{i}2\pi} u'(x).$$

Quantization examples

For example, the Wigner transform $C = W$ from formula (15) gives rise to the *Weyl quantization* $a \mapsto a_W$:

$$a_W v(x) = \int_{\widehat{\mathbb{R}}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(x-y)\cdot\eta}\, a(\frac{x+y}{2}, \eta)\, v(y)\, \mathrm{d}y\, \mathrm{d}\eta. \qquad (22)$$

Born–Jordan transform Q gives Born–Jordan quantization $a \mapsto a_Q$,

$$a_Q v(x) = \int_{\widehat{\mathbb{R}}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(x-y)\cdot\eta}\, \frac{1}{y-x} \int_x^y a(t,\eta)\, \mathrm{d}t\, v(y)\, \mathrm{d}y\, \mathrm{d}\eta. \qquad (23)$$

What makes the Born–Jordan quantization unique among quantizations is that if $a(x,\eta) = f(x)$ and $b(x,\eta) = \widehat{g}(\eta)$ then

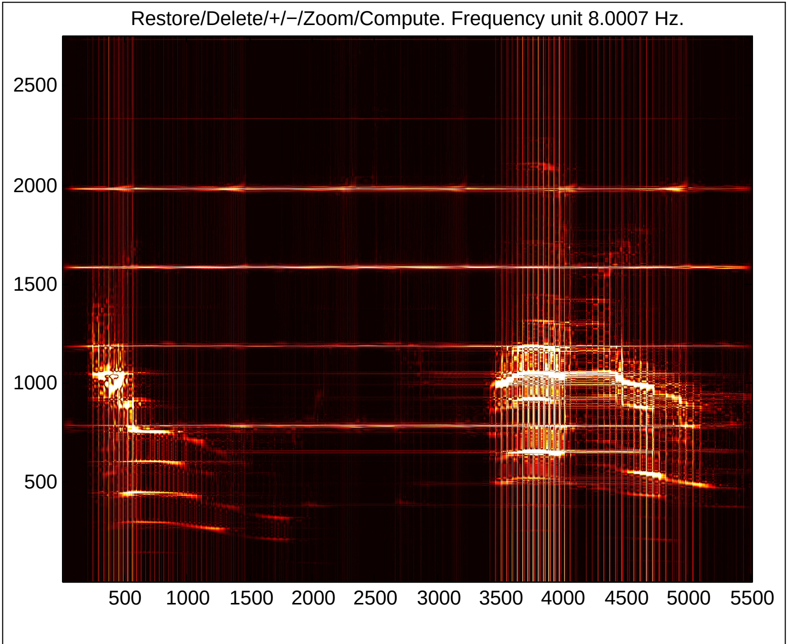
$$\{a,b\}_Q = -\mathrm{i}2\pi\, [a_Q, b_Q], \qquad (24)$$

where the Poisson bracket $\{a,b\}$ is reduced to

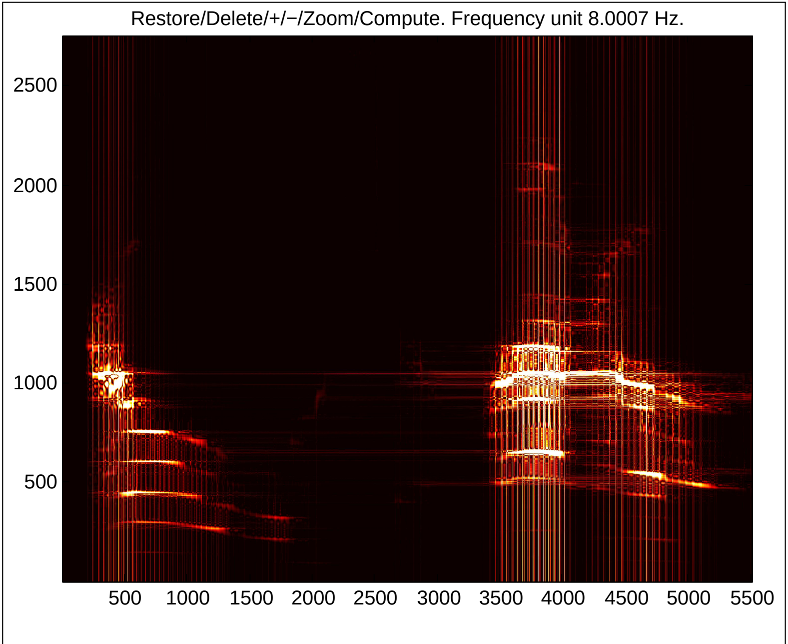
$$\{a,b\}(x,\eta) = \left(\frac{\partial a}{\partial x}\frac{\partial b}{\partial \eta} - \frac{\partial a}{\partial \eta}\frac{\partial b}{\partial x}\right)(x,\eta) = f'(x)\, \widehat{g}'(\eta). \qquad (25)$$

(24)-remark: no-go-theorems [Groenewold 1946, van Hove 1951].

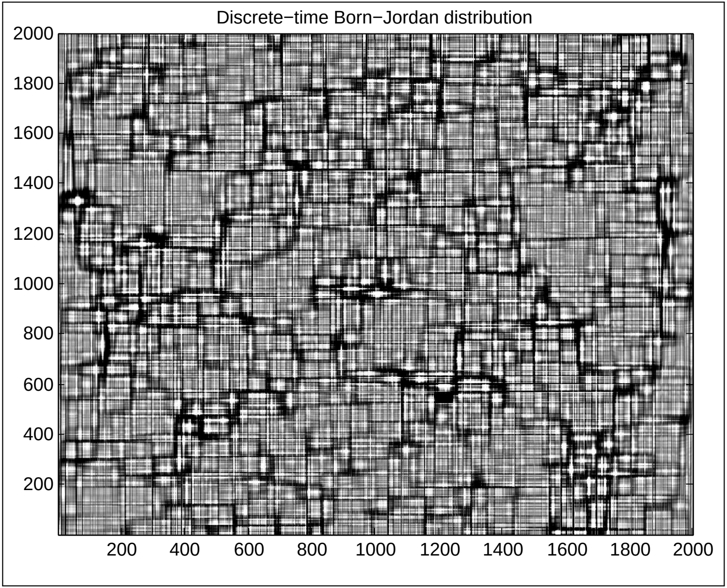
MRI + "patient" (frequency unit 1 Hz)



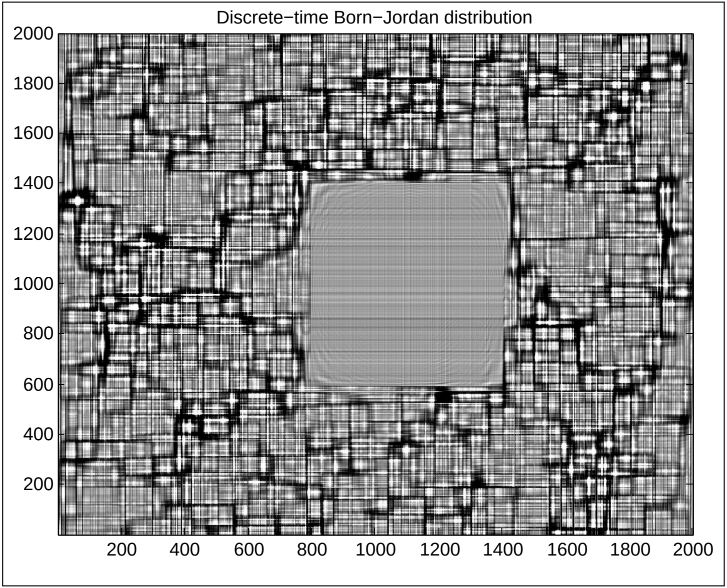
MRI + "patient" after noise filtering



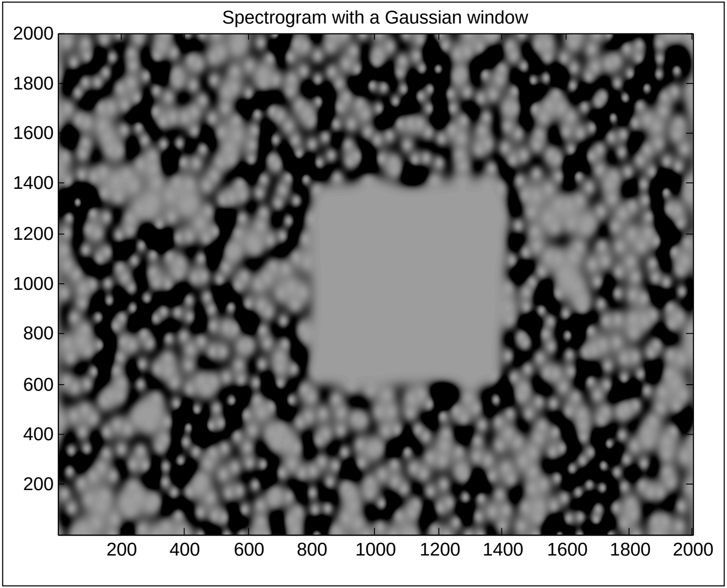
Born–Jordan energy density of noise and...



… and localized noise (complement of a rectangle)



… and a spectrogram of same Born–Jordan -localized noise



Relation of different quantizations

Let $C = \psi * W$. Then

$$\begin{aligned} \langle u, a_C v\rangle &= \langle C(u,v), a\rangle \\ &= \langle \psi * W(u,v), a\rangle \\ &= \langle (F\psi)\, Fw(u,v), Fa\rangle \\ &= \langle Fw(u,v), \overline{(F\psi)}\, Fa\rangle \\ &= \langle W(u,v), b\rangle \\ &= \langle u, b_W v\rangle, \end{aligned}$$

where $Fb = \overline{(F\psi)}\, Fa$. That is, $a_C = b_W$ here.
Quantization $a \mapsto a_C$ is surjective if $[u] \mapsto C[u]$ is invertible.
Quantization $a \mapsto a_C$ is injective if $F\psi$ does not vanish.

Examples: Weyl and Kohn–Nirenberg quantizations are bijective:

$$a(x,\eta) = \mathrm{e}^{-\mathrm{i}2\pi x\cdot\eta}\, a_{KN}(x \mapsto e^{\mathrm{i}2\pi x\cdot\eta}).$$

Born-Jordan $[u] \mapsto Q[u] = \psi * W[u]$ is invertible, with
$F\psi(\xi, y) = \mathrm{sinc}(\xi \cdot y)$. Here $a \mapsto a_Q$ is surjective but not injective.

Orthonormal bases have uniform energy densities

Theorem. *Let* $\{u_k\}_{k=1}^{\infty}$ *be an orthonormal basis of* $L^2(\mathbb{R})$*.*
Let $C = \psi * W$ *with energy normalization* $\iint \psi(x,\eta)\,\mathrm{d}x\,\mathrm{d}\eta = 1$*.*
Then for almost every $(x,\eta) \in \mathbb{R} \times \widehat{\mathbb{R}}$

$$\sum_{k=1}^{\infty} C[u_k](x,\eta) = 1$$

(with easy generalization to tight frames and non-normalized C.)

Proof. Let $a(x,\eta) := \sum_{k=1}^{\infty} W[u_k](x,\eta)$. Here $a_W v = v$, because

$$\begin{aligned}\langle v, a_W v\rangle = \langle W(v,v), a\rangle &= \sum_{k=1}^{\infty}\langle W[v], W[u_k]\rangle \\ &\overset{Moyal}{=} \sum_{k=1}^{\infty} |\langle v, u_k\rangle|^2 \overset{Parseval}{=} \langle v, v\rangle.\end{aligned}$$

Thereby $a(x,\eta) = 1$ for almost every $(x,\eta) \in \mathbb{R} \times \widehat{\mathbb{R}}$. Hence
$\sum_{k=1}^{\infty} C[u_k](x,\eta) = \psi * a(x,\eta) = \iint \psi(t,\omega)\,\mathrm{d}t\,\mathrm{d}\omega = 1.$ □

On Born–Jordan boundedness

Let $a = f \otimes \widehat{g}$, where f continuous bounded ($|f| \leq \|f\|_{L^\infty} < \infty$), $g \geq 0$ integrable ($\int g(x)\,\mathrm{d}x = \widehat{g}(0) < \infty$, so $|\widehat{g}| \leq \widehat{g}(0) = \|\widehat{g}\|_{L^\infty}$). Then

$$\begin{aligned} \|a_Q(X,D)v\|^2 &= \int \left| \int g(x-y)\,\frac{1}{y-x}\int_x^y f(t)\,\mathrm{d}t\,v(y)\,\mathrm{d}y \right|^2 \mathrm{d}x \\ &\leq \|f\|_{L^\infty}^2 \int \left[\int g(x-y)\,|v(y)|\,\mathrm{d}y \right]^2 \mathrm{d}x \\ &\leq \|f\|_{L^\infty}^2 \|\widehat{g}\|_{L^\infty}^2 \|v\|^2 \\ &= \|a\|_{L^\infty}^2 \|v\|^2 \end{aligned}$$

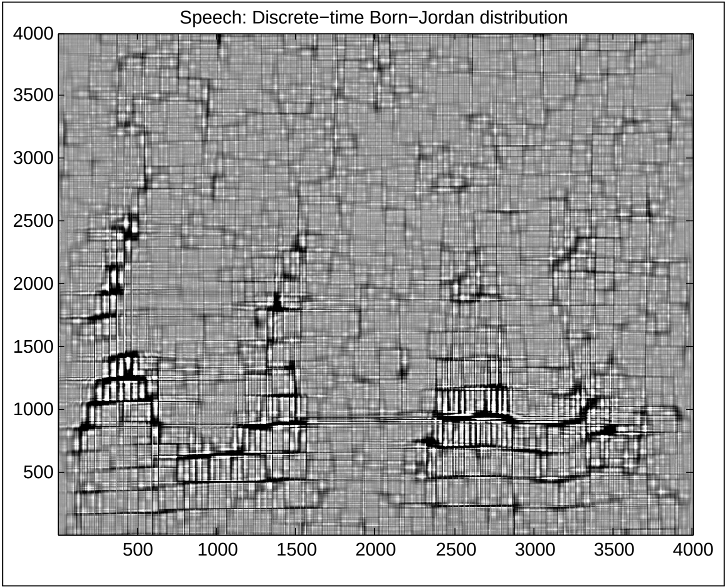
so for all $v \in \mathscr{H} = L^2(\mathbb{R})$, we obtain the norm bound

$$\|a_Q(X,D)v\| \leq \|v\| \max_{(t,\eta)\in\mathbb{R}\times\widehat{\mathbb{R}}} |a(t,\eta)|.$$

For any $a \in L^1(\mathbb{R} \times \widehat{\mathbb{R}})$ we have the norm bound

$$\|a_Q v\| \leq \pi\,\|a\|_{L^1}\,\|v\|.$$

Born–Jordan of noisy speech...



… enhanced after two localizations