CS-E5755 Nonlinear Dynamics and Chaos
Exam 9.9.2021
Open Book Exam
Problems 1, 2, 4, and 5 are pen&paper problems, problem 3 is numerical.

Problem 1. (6 p) Consider the system $\dot{x} = rx - \sinh x$. Sketch the bifurcation diagram. Show calculations for everything you need to know in order to draw bifurcation diagram correctly.

Problem 2 (6 p) Consider the system $\ddot{x} = x^2 - x$. Find the fixed (equilibrium) points and classify them. Sketch the phase portrait. You must justify the phase portrait - everything it includes.
(You are not expected to plot anything numerically, but if you should do that for checking, please do not attach it to your solution - no points available for that.)

Problem 3. (6 p) Plot **numerically** the phase portraits of the following two systems to determine what kind of a bifurcation takes place in each case. In the phase portraits you can limit your interest in an appropriate range around the origin. There may or may not be something else going on elsewhere in the phase plane, but you only need to determine the bifurcation that is seen in a phase plane area within which the origin lies. (No pen and paper calculations are necessary here.)

a) $\dot{x} = y + \mu x, \dot{y} = -x + \mu y - x^2 y$. Bifurcation at $\mu = 0$.
b) $\dot{x} = y + \mu x - x^2, \dot{y} = -x + \mu y - 2x^2$. Bifurcation at $\mu = 0$.

Note: Indicate everything that belongs to the phase portrait one way or another; that is, if numerical plots lack some information, complete the plots, for example, with descriptive text. You must give proper arguments for determining the bifurcation.
Hint: If your numerical integration takes long for some initial values, then interrupt (the kernel) and sketch the result for a smaller number of time steps.

Problem 4. (6 p) Find all values of r at which the logistic map $x_{n+1} = rx_n(1 - x_n)$, where $0 \leq x_n \leq 1$ and $0 \leq r \leq 4$, has a superstable fixed point.

Problem 5. Are the following statements true or false? Justify your answers.

a) (1 p) The distance of two points (x, y, z) initially separated by a small distance and moving according to chaotic dynamics described by the Lorenz system will grow indefinitely (forever).
b) (1 p) One could in principle construct a Lorenz map for the attractor of the Rössler system.
(Continued on the next page.)

c) (1 p) Attractors can only exist for dissipative dynamics, that is, for dynamics where energy is dissipated for example through friction.