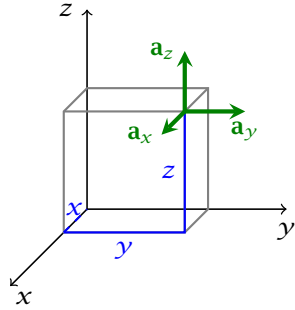# Nabla operations

Cartesian coordinates (x, y, z)

$$\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$
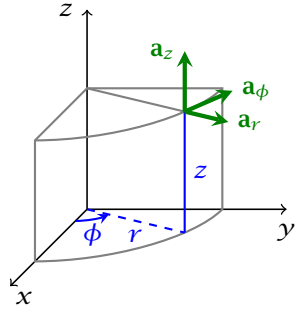
Cylindrical coordinates (r, ϕ, z)

$$\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\phi \frac{1}{r} \frac{\partial V}{\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$\nabla \times \mathbf{A} = \frac{1}{r} \begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & r A_\phi & A_z \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} \left(r A_r \right) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$
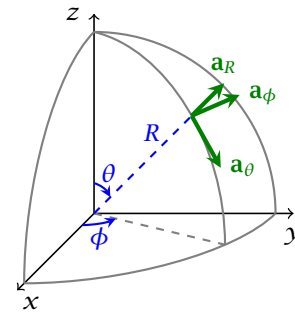
Spherical coordinates (R, θ, ϕ)

$$\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \mathbf{a}_\theta \frac{1}{R} \frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin \theta} \begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R \sin \theta \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_R & R A_\theta & (R \sin \theta) A_\phi \end{vmatrix}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 A_R \right) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} \left(A_\theta \sin \theta \right) + \frac{1}{R \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$$

Coordinate transformations

Cartesian ↔ Cylindrical

$$x = r \cos \phi, \qquad y = r \sin \phi, \qquad z = z$$

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1} \frac{y}{x}, \qquad z = z$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cartesian ↔ Spherical

$$x = R \sin \theta \cos \phi, \qquad y = R \sin \theta \sin \phi, \qquad z = R \cos \theta$$

$$R = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}, \quad \phi = \tan^{-1} \frac{y}{x}$$

$$\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Cylindrical ↔ Spherical

$$r = R \sin \theta, \qquad \phi = \phi, \qquad z = R \cos \theta$$

$$R = \sqrt{r^2 + z^2}, \qquad \theta = \tan^{-1} \frac{r}{z}, \qquad \phi = \phi$$

$$\begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix} = \begin{pmatrix} \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix}$$

$$\begin{pmatrix} A_R \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} \sin \theta & 0 & \cos \theta \\ \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} A_r \\ A_\phi \\ A_z \end{pmatrix}$$

Other useful formulas

Cartesian coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_x \, dx + \mathbf{a}_y \, dy + \mathbf{a}_z \, dz$$
$$ds_x = dy \, dz$$
$$ds_y = dx \, dz$$
$$ds_z = dx \, dy$$
$$dv = dx \, dy \, dz$$

Cylindrical coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_r \, dr + \mathbf{a}_\phi r \, d\phi + \mathbf{a}_z \, dz$$
$$ds_r = r \, d\phi \, dz$$
$$ds_\phi = dr \, dz$$
$$ds_z = r \, dr \, d\phi$$
$$dv = r \, dr \, d\phi \, dz$$

Spherical coordinates

$$d\boldsymbol{\ell} = \mathbf{a}_R \, dR + \mathbf{a}_\theta R \, d\theta + \mathbf{a}_\phi R \sin \theta \, d\phi$$
$$ds_R = R^2 \sin \theta \, d\theta \, d\phi$$
$$ds_\theta = R \sin \theta \, dR \, d\phi$$
$$ds_\phi = R \, dR \, d\theta$$
$$dv = R^2 \sin \theta \, dR \, d\theta \, d\phi$$

Divergence theorem $\int_V \nabla \cdot \mathbf{A} \, dv = \oint_S \mathbf{A} \cdot d\mathbf{s}$

Stokes' theorem $\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}$

Constants

$$c = 299\,792\,458 \, \frac{\text{m}}{\text{s}}$$

$$\mu_0 = 4\pi \times 10^{-7} \, \frac{\text{Vs}}{\text{Am}} \approx 1.257 \times 10^{-6} \, \frac{\text{H}}{\text{m}}$$

$$\varepsilon_0 = \frac{1}{\mu_0 \, c^2} \approx 8.854 \times 10^{-12} \, \frac{\text{As}}{\text{Vm}} \quad \left(= \frac{\text{F}}{\text{m}} \right)$$

$$e \approx 1.602 \times 10^{-19} \, \text{C}$$